Weighted composition operators on Dirichlet space

Isabelle Chalendar

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Joint work with Eva Gallardo (Madrid, Spain) and Jonathan Partington (Leeds, UK)

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$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z).$$

First remarks

$$\mathcal{D} \subset H^2 \subset \mathcal{B}$$

where H^2 is the **Hardy** space,

$$H^2 := \{f : \mathbb{D} \to \mathbb{C} \text{ analytic}, \ \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{it})|^2 dt < \infty\}$$

and $\mathcal{B}$ is the **Bergman** space,

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A Primer on the Dirichlet Space, Cambridge Tracts in Mathematics
O. El-Fallah, K. Kellay, J. Mashreghi, T. Ransford, 2014.

Bounded composition operators

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Moreover $\varphi : \mathbb{D} \to \mathbb{D}$ analytic does not imply $\varphi \in \mathcal{D}$, e.g **infinite** Blaschke products are never in $\mathcal{D}$.
In 1980, Voas (Phd thesis) characterized $\varphi \in \mathcal{D}$ such that C_φ is bounded on $\mathcal{D}$.

Weighted composition operator

Let $u \in \mathcal{D}$, $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, and define the **weighted composition operator**

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If C_φ is bounded on $\mathcal{D}$, if u is a **multiplier** of $\mathcal{D}$ (i.e. $T_u(f) := uf$ is a bounded operator on $\mathcal{D}$), then $W_{u,\varphi}$ is also bounded on $\mathcal{D}$.

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Definition: μ is a **Carleson measure** for $\mathcal{D}$ if $\mathcal{D}$ injects continuously into $L^2(\mathbb{D}, \mu)$.
Therefore u is a multiplier for $\mathcal{D}$ iff $d\mu(z) = |u'(z)|^2 dA(z)$ is a Carleson measure for $\mathcal{D}$.

An example

Let $u(z) = (1-z)^2$ and let φ be the infinite Blaschke product with zeroes $(1-1/n^2)_{n\geq 1}$.

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However, for $f \in \mathcal{D}$,

$$((1-z)^2(f \circ \varphi))' = -2(1-z)(f \circ \varphi) + (1-z)^2(f' \circ \varphi)\varphi'.$$

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The first term is in $\mathcal{B}$, $(1-z)^2\varphi'$ is bounded, $f' \circ \varphi$ is in $\mathcal{B}$.
Therefore $W_{u,\varphi}$ is bounded on $\mathcal{D}$...and C_φ is **not**.

Space of multipliers

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u is in $\mathcal{D}$ but is not a multiplier, and $W_{u,\varphi}$ is bounded.

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Theorem 2: Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. Then $\mathcal{M}(\varphi) = \mathcal{D}$ iff $\|\varphi\|_\infty < 1$ and φ is a multiplier of $\mathcal{D}$.

Analytic functions on the disc $\mathbb{D}$

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in the unit disc $\mathbb{D}$ that have finite norm

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So $\|z^n\| = \beta_n$.

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The case $\beta_n = \sqrt{n+1}$ provides the **Dirichlet** space, which is included in H^2.

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General boundedness conditions are known but are more complicated (e.g. in terms of Carleson measures or reproducing kernels).

Lower-triangular operators

Theorem 3: Suppose $\mathcal{X}$ is a complex Banach space with a 1-unconditional basis $(x_j)_{j\geq 0}$ (e.g. ℓ^p spaces), and that $A : \mathcal{X} \to \mathcal{X}$ is bounded and lower-triangular w.r.t. this basis.

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Namely let $\Omega(z) = D^{-z}AD^{z}$ and use Phragmén-Lindelöf to show that $\|\Omega(z)\|$ attains its maximum on the imaginary axis.

Consequences for $H^2(\beta)$

Corollary 4: Let T be a bounded operator on H^2 given by a lower-triangular matrix with respect to the basis $(z^n)_{n\geq 0}$.

Take $(\beta_n)_{n\geq 0}$ positive and **decreasing** (e.g. Bergman space).

Then T is bounded on $H^2(\beta)$ and

$$\|T\|_{H^2(\beta)} \leq \|T\|_{H^2}.$$

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Known special cases:
C_φ (Cowen–MacCluer, using Hadamard–Schur products);
T_h (can be deduced from Kac'nelson's lemma).

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It is known that C_ψ is bounded (see Gallardo-Gutiérrez–Partington for recent norm estimates).

Compose with this to get the general result

$$\|W_{h,\varphi}\|_{H^2(\beta)} \leq \|C_\psi\|_{H^2(\beta)} \|W_{h,\varphi}\|_{H^2} \|C_\psi\|_{H^2}.$$

Consequences for H^2

Corollary 5: Let T be a bounded operator on $H^2(\beta)$ given by a lower-triangular matrix with respect to the basis $(z^n)_{n\geq 0}$.

Take $(\beta_n)_{n\geq 0}$ positive and **increasing** (e.g. Dirichlet space).

Then T is bounded on H^2 and

$$\|T\|_{H^2} \leq \|T\|_{H^2(\beta)}.$$

Paley–Wiener theorems

Recall that the Laplace transform

$$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}\,dt$$

provides an isomorphism of $L^2(0,\infty)$ onto $H^2(\mathbb{C}_+)$, the Hardy space of the right half-plane, with

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A wider class of spaces and isometric mappings was introduced by Zen Harper (2006), extended by Jacob–Partington–Pott (2013).

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Examples:
$\nu = \delta_0$ (Dirac), $w(t) \equiv 2\pi$.

Measures defining Zen spaces

Let ν be a (not necessarily finite) positive Borel measure on $(0,\infty)$ with $\nu[0,2x] \leq C\nu[0,x)$ for all $x>0$.Let

$$w(t) = 2\pi \int_0^\infty e^{-2xt}\, d\nu(x) \qquad (t \geq 0).$$

Examples:
$\nu = \delta_0$ (Dirac), $w(t) \equiv 2\pi$.
$d\nu(x) = dx$ (Lebesgue), $w(t) = \pi/t$.

The Laplace isometry

Then $\mathcal{L}$ is an isometric mapping from $L^2(0,\infty; w(t)\,dt)$ into the space A^2_ν of analytic functions f on $\mathbb{C}_+$ whose norm

$$\|f\| = \left(\int_0^\infty \int_{-\infty}^\infty |f(x+iy)|^2\,d\nu(x)\,dy\right)^{1/2}$$

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$L^2(0,\infty; dt/t)$ gives the Bergman space, etc.

Causal operators

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Theorem. If $A : L^2(0, \infty) \to L^2(0, \infty)$ is causal and bounded, then for every **decreasing** weight w it is bounded on $L^2(0, \infty; w(t)\, dt)$ with

$$\|A\|_{L^2(0,\infty; w(t)\, dt)} \le \|A\|_{L^2(0,\infty)}.$$

Weighted composition operators on the half-plane

If $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$ is analytic, then it has the Nevanlinna representation

$$\varphi(s) = as + ib + \int_{-\infty}^{\infty} \frac{1 - its}{s - it} \, d\mu(t),$$

with $a \geq 0$, $b \in \mathbb{R}$ and μ a measure of $\mathbb{R}$.

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Now $W_{h,\varphi}$ will be causal if h is analytic and $a \geq 1$. For the

non-causal case we can again compose with an automorphism to get a norm estimate.

The theorem for weighted composition operators

Theorem. Let $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$ be analytic with $a > 0$ the number occurring in its Nevanlinna representation, and let $h : \mathbb{C}_+ \to \mathbb{C}$ be holomorphic such that $W_{h,\varphi}$ is bounded on $H^2(\mathbb{C}_+)$.

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Theorem. Let $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$ be analytic with $a > 0$ the number occurring in its Nevanlinna representation, and let $h : \mathbb{C}_+ \to \mathbb{C}$ be holomorphic such that $W_{h,\varphi}$ is bounded on $H^2(\mathbb{C}_+)$. Then $W_{h,\varphi}$ is also bounded on A^2_ν and

$$\|W_{h,\varphi}\|_{A^2_\nu} \le \|C_\psi\|_{A^2_\nu} \|W_{h,\varphi}\|_{H^2(\mathbb{C}_+)} \|C_\psi^{-1}\|_{H^2(\mathbb{C}_+)},$$

where $\psi(s) = as$.

Some exact answers for composition operators

Elliott–Jury (2012): C_φ is bounded on $H^2(\mathbb{C}_+)$ if and only if

$$\text{n.t.lim}_{z\to\infty}\varphi(z) = \infty,$$

and

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Now for causality $0 < \lambda \leq 1$ and, as expected, $\|C_\varphi\|_{A^2_\nu} \leq \|C_\varphi\|_{H^2}$.

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