Arveson-Douglas conjecture and Toeplitz operators

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September 3 - 5, 2014
Queen's University Belfast

International Workshop on Operator Theory

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Drury-Arveson space: Definition

Let $\mathbb{B} = \{z \in \mathbb{C}^d;\ |z| < 1\}$ be the unit ball in $\mathbb{C}^d$. The Drury-Arveson space

$$H^2_d = \{f = \sum_\nu f_\nu z^\nu;\ \|f\|^2 = \sum_\nu \frac{\nu!}{|\nu|!}|f_\nu|^2 < \infty\} \subset \mathcal{O}(\mathbb{B})$$

is an analytic functional Hilbert space with reproducing kernel

$$K : \mathbb{B} \times \mathbb{B} \to \mathbb{C}, K(z,w) = \frac{1}{1 - \langle z, w\rangle}.$$

The multiplication tuple $M_z = (M_{z_1}, \dots, M_{z_d}) \in L(H^2_d)^d$ is a row contraction

$$\sum_{i=1}^d M_{z_i} M_{z_i}^* \leq 1.$$

Many classical results extend to the higher dimensional case when the Hardy space

$$H^2(\mathbb{B}_d) = H(\frac{1}{(1 - \langle z, w\rangle)^d}) \quad (\overset{d=1}{\equiv} H^2_1)$$

is replaced by the Drury-Arveson space.

Drury-Arveson space: Typical results

von Neuman inequality:

- For $d \geq 3$ there are commuting contractions $T \in L(H)^d$ and polynomials $p \in \mathbb{C}[z]$ such that

$$\|p(T)\| > \|p\|_{\mathbb{D}^d} \qquad \text{(Varopoulos '74, Crabb – Davie '75)}$$

- For $d \geq 2$ there are a row contraction $T \in L(H)^d$ and polynomials $(p_k) \in \mathbb{C}[z]$ with

$$\|p_k\|_{\mathbb{B}} \leq 1, \quad \text{but} \quad \|p_k(T)\| \stackrel{k}{\to} \infty \qquad \text{(Drury '78)}$$

- For $M_z \in L(H^2_d)^d$ and any row contraction $T \in L(H)^d$ (Drury '78, Arveson '98)

$$\|p(T)\| \leq \|p(M_z)\| = \|M_p\| = \|p\|_{\mathcal{M}} \quad \forall p \in \mathbb{C}[z]^{n,n}, n \geq 1.$$

Drury-Arveson space: Typical results

Nevanlinna-Pick interpolation: For $z_1, \ldots, z_k \in \mathbb{D}$, $w_1, \ldots, w_k \in \mathbb{C}$:

$$\exists f \in H^\infty(\mathbb{D}) \text{ with } \|f\|_{\mathbb{D}} \leq 1 \text{ and } f(z_i) = w_i \ \forall i = 1, \ldots, k$$

$$\Leftrightarrow \quad \left(\frac{1 - w_i \overline{w}_j}{1 - z_i \overline{z}_j}\right)_{i,j} \in \mathbb{C}_+^{k,k}$$

There is no direct generalization to dimensions $d > 1$.

However, one can show that for $z_1, \ldots, z_k \in \mathbb{B}$, $w_1, \ldots, w_k \in \mathbb{C}$:

$$\exists f \in \mathcal{M}(H_d^2) \text{ with } \|M_f\| \leq 1 \text{ and } f(z_i) = w_i \ \forall i = 1, \ldots, k$$

$$\Leftrightarrow \quad \left(\frac{1 - w_i \overline{w}_j}{1 - \langle z_i, z_j \rangle}\right)_{i,j} \in \mathbb{C}_+^{k,k}$$

Drury-Arveson space: Typical results

Model theory (Sz.-Nagy-Foias, Müller-Vasilescu, Arveson, ...)

Let $T \in L(H)^d$ be a row contraction ($\sum T_i T_i^* \leq 1_H$). Define

$$D = (1 - \sum T_i T_i^*)^{1/2} \quad \text{defect operator}, \quad \mathcal{D} = \overline{DH} \quad \text{defect space}.$$

Then

$$T = P_H[(M_z \otimes 1_{\mathcal{D}}) \oplus W]_{|H} \in L((H_d^2 \otimes \mathcal{D}) \oplus K)$$

with a spherical unitary $W \in L(K)^d$, that is, a commuting tuple of normal operators with

$$\sum W_i W_i^* = 1_K.$$

Operator theory on projective varieties

For homogeneous polynomials $p_1, \dots, p_r \in \mathbb{C}[z]$, is there a universal solution S of the operator equations

$$p_i(T) = 0 \quad (1 \le i \le r)$$

in the class of all row contractions? We would like that, for all such T,

- $\|p(T)\| \le \|p(S)\| \quad \forall p \in \mathbb{C}[z]$
- $T = P_H(S \otimes 1_{\mathcal{D}}) \oplus W_{|H} \quad \text{(with a spherical unitary } W)$

Idea: Define $M = \overline{(p_1, \dots, p_r)} \subset H_d^2$ and $S = M_z/M \cong P_{M^\perp} M_z|_{M^\perp}$

$$\Rightarrow p(S) = p(M_z)/M = 0 \quad \forall p \in I = (p_1, \dots, p_r)$$

Essential normality

The tuple $M_z \in L(H^2_d)^d$ is Fredholm and

$$[M_z] = ([M_{z_1}], \dots, [M_{z_d}]) \in \mathcal{C}(H^2_d)^d$$

is a spherical unitary in the Calkin algebra $\mathcal{C}(H^2_d) = L(H^2_d)/\mathcal{K}(H^2_d)$. In particular,

$$\exists \quad 0 \to \mathcal{K}(H^2_d) \hookrightarrow C^*(M_z) \to C(\partial\mathbb{B}) \to 0 \quad \text{exact sequence of } C^*\text{–algebras}$$

Problem: Is $S = P_{M^\perp} M_z|_{M^\perp}$ Fredholm, an essential spherical unitary? Since

$$\sum_{1=1}^{d} S_i S_i^* = P_{M^\perp}(\sum_{1=1}^{d} M_{z_i} M_{z_i}^*)|_{M^\perp} \in I + \mathcal{K}(M^\perp),$$

it suffices to prove that S is essentially normal, that is,

$$[S_i, S_i^*] \in \mathcal{K}(M^\perp) \quad (i = 1, \dots, d).$$

Arveson conjecture

Arveson: $M_z \in L(H^2_d)^d$ is p-essentially normal for every $p > d$, that is, for $p > d$

$$[M_{z_j}, M^*_{z_k}] \in \mathcal{S}^p \quad \text{(Schatten class)}.$$

Conjecture: $M = \bar{I} \Rightarrow S = M_z/M \cong P_{M^\perp} M_z|_{\mathcal{M}^\perp}$ is p-essentially normal for $p > d$

$M_z \in L(H^2_d)^d$ is graded: $H^2_d = \oplus_{k=0}^{\infty} \mathbb{H}_k$ (homogeneous poly's of degree k) and

$$M_{z_i}\mathbb{H}_k \subset \mathbb{H}_{k+1}, \quad H^2_d = \bigvee_{\alpha} M_z^{\alpha}\mathbb{H}_0, \quad \sum_{i=1}^{d} M_{z_i} H^2_d \subset H^2_d \text{ closed}$$

Let N be the number operator on H^2_d. More generally, for $f : \mathbb{N} \to \mathbb{R}$,

$$f(N) : D_f \to H^2_d, \; h = \sum_{k=0}^{\infty} h_k \mapsto \sum_{k=0}^{\infty} f(k)\, h_k$$

defines a closed operator with dense domain $\mathbb{C}[z] \subset D_f \subset H^2_d$.

Number operator

Arveson: $[M_{z_j}, M_{z_k}^*] = (N+1)^{-1}(\delta_{jk} - M_{z_k} M_{z_j}^*) \in \mathcal{S}^p$ for $p > d$, since

$$\mathrm{tr}(N+1)^{-p} = \sum_{k=0}^{\infty} \frac{\dim \mathbb{H}_k}{(k+1)^p} = \sum_{k=0}^{\infty} \frac{r(k)}{(k+1)^p} < \infty \text{ for } p > d,$$

where $r \in \mathbb{Q}[x]$ with $\deg(r) = d - 1$

$M \in \mathrm{Lat}(M_z)$ homogeneous if $M = \bigvee_{k=0}^{\infty} M \cap \mathbb{H}_k \Leftrightarrow \exists p = (p_1, \ldots, p_r)$ $(p_i \in \mathbb{H}_{k_i})$ with

$$M = \overline{(p_1, \ldots, p_r)}$$

$\Rightarrow S = M_z/M = P_{M^\perp} M_z|_{M^\perp} \in L(M^\perp)$ graded wrt $M^\perp = \bigvee_{k=0}^{\infty} M^\perp \cap \mathbb{H}_k$

$$\mathrm{tr}(N+1)^{-p}|_{M^\perp} = \sum_{k=0}^{\infty} \frac{\dim(M^\perp \cap \mathbb{H}_k)}{(k+1)^p} = \sum_{k=0}^{\infty} \frac{r_M(k)}{(k+1)^p} < \infty \text{ for } p > n = \dim_0 Z(p),$$

where $r_M(k)$ for large k is given by a (Hilbert) polynomial of degree $n - 1$.

Arveson-Douglas conjecture

Arveson-Douglas conjecture:

Let $M = \overline{(p_1, \ldots, p_r)}$ be homogeneous, $S = M_z/M \cong P_{M^\perp} M_z|_{M^\perp} \in L(M^\perp)^d$.

Show that: $[S_j, S_k^*] \in \mathcal{S}^p \quad \text{for all } p > n = \dim_0 Z(p)$.

Known results:

- Arveson '05, Douglas '06 $\quad p_i = z^{\alpha_i}$
- Guo-Wang '08 $\quad d \leq 3$ or $r = 1$ (principal ideal case)
- Kennedy-Shalit '12 $\quad \mathrm{span}(Z(p_i)) \cap \mathrm{span}(Z(p_j)) = \{0\}$ for $i \neq j$.

Hilbert's Nullstellensatz

$I \lhd \mathbb{C}[z]$ homogeneous , $Z(I)$ common zero set, $\sqrt{I} = \{p \in \mathbb{C}[z]; \exists k : p^k \in I\}$

Hilbert's Nullstellensatz: $I = \{p \in \mathbb{C}[z]; p|_{Z(I)} \equiv 0\} \Leftrightarrow I = \sqrt{I}$ ($\overset{def}{\Leftrightarrow}$ I radical)

$\Rightarrow$ For $M \in \mathrm{Lat}(M_z)$ the following are equivalent:

(i) $\exists V \subset \mathbb{C}^d$ homogeneous (that is, $tV \subset V$ for all $t \in \mathbb{C}$) with

$$M = \{f \in H^2_d; f|_{V \cap \mathbb{B}} \equiv 0\}$$

(ii) $\exists I \lhd \mathbb{C}[z]$ homogeneous radical with $M = \bar{I}$

Geometric Arveson-Douglas conjecture: $M = \bar{I}$ for a homogeneous radical ideal

Then: $[S_j, S_k^*] \in \mathcal{S}^p$ for all $p > \dim_0 Z(I)$

Analytic Besov-Sobolev spaces

For $\alpha > -(d+1)$ the generalized Bergman spaces

$$A^2_\alpha = \{f = \sum_\nu f_\nu z^\nu;\ \|f\|^2_\alpha = \sum_\nu \frac{\nu!}{|\nu|!} \frac{|\nu|!\Gamma(d+\alpha+1)}{\Gamma(d+\alpha+1+|\nu|)} |f_\nu|^2 < \infty\} \subset \mathcal{O}(\mathbb{B})$$

are analytic functional Hilbert spaces on $\mathbb{B}$ with reproducing kernel

$$K_\alpha(z,w) = \frac{1}{(1-\langle z,w\rangle)^{d+\alpha+1}}.$$

In particular: $A^2_{-d} = H^2_d$, $\quad A^2_{-1} = H^2(\mathbb{B})$ (Hardy space), $A^2_0 = L^2_a(\mathbb{B})$ (Bergman space)

The equivalent norms (Stirling's formula)

$$\|f\|^2_{\alpha\circ} = \sum_\nu \frac{\nu!}{|\nu|!} \frac{1}{(|\nu|+1)^{d+\alpha}} |f_\nu|^2$$

extend the definition to all $\alpha \in \mathbb{R}$. In fact (as vector spaces)

$$A^2_{\alpha\circ} = W^{-\alpha/2}_{\text{hol}}(\mathbb{B}) \quad \text{(Beatrous and Burbea '89)}$$

Main result

Let A^2_α $(\alpha \in \mathbb{R})$ be the space $A^2_{\alpha\circ}$ equipped with the norm

$$\|\cdot\|_\alpha \text{ or } \|\cdot\|_{\alpha\circ} \quad \text{for} \quad \alpha > -(d+1), \quad \|\cdot\|_{\alpha\circ} \quad \text{for} \quad \alpha \leq -(d+1)$$

Then as above one can show that for all $\alpha \in \mathbb{R}$

$$M_z \in L(A^2_\alpha)^d \text{ is } p\text{−essentially normal for all } p > d.$$

Theorem (Englis-E.)

Let $V \subset \mathbb{C}^d$ be a homogeneous variety smooth in every $p \in V \setminus \{0\}$, let $\alpha \in \mathbb{R}$ and

$$M = \{f \in A^2_\alpha;\ f|_{V \cap \mathbb{B}} \equiv 0\}.$$

Then $S = P_{M^\perp} M_z|_{M^\perp}$ is p-essentially normal for every $p > \dim_0(V)$.

Pseudodifferential operators (ΨDO)

A ΨDO of order $m \in \mathbb{R}$ is an operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ of the form

$$Au(x) = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi)\hat{u}(\xi)\,\mathrm{d}\xi \qquad \text{with } a \in C^\infty \text{ such that}$$

$$\forall \alpha, \beta\ \exists C_{\alpha\beta} : \quad |D_x^\beta D_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta}(1 + |\xi|)^{m-|\alpha|} \quad (x, \xi \in \mathbb{R}^n)$$

For $\Psi^m = \{\Psi\text{DO}'\text{s of order } m\}$, $\Psi^m\Psi^k \subset \Psi^{m+k}$ and (Sobolev mapping properties)

$$\text{each } A \in \Psi^m \text{ induces bounded operators } W^s(\mathbb{R}^n) \to W^{s-m}(\mathbb{R}^n) \quad (s \in \mathbb{R}),$$

where

$$W^s(\mathbb{R}^n) = (\mathcal{S}(\mathbb{R}^n), \|u\| = (\int (1 + |\xi|^2)^s |u|^2\ \mathrm{d}\xi)^{1/2})^\sim \text{ completion}$$

Everthing makes sense on compact C^∞-manifolds M with $\mathcal{S}(\mathbb{R}^n)$ replaced by $C^\infty(M)$.

Generalized Toeplitz operators (Boutet de Monvel and Guillemin)

$M = \partial\Omega, \Omega \subset \mathbb{C}^n$ (or n-dim. complex mfd) smooth strictly pseudoconvex domain

$(W^s(\partial\Omega))_{s\in\mathbb{R}}$ Sobolev spaces, $W^0(\partial\Omega) = L^2(\partial\Omega)$, $C^\infty(\partial\Omega) \subset W^s(\partial\Omega)$ dense

$$W^s_{\mathrm{hol}}(\partial\Omega) = \overline{C^\infty_{\mathrm{hol}}(\bar{\Omega})}|_{\partial\Omega} \subset W^s(\partial\Omega),\ \Pi :\ W^s(\partial\Omega) \to W^s_{\mathrm{hol}}(\partial\Omega)\ \text{Szego projection}$$

Generalized Toeplitz operators (GTO's) are Toeplitz operators

$$T_Q : W^s_{\mathrm{hol}}(\partial\Omega) \to W^{s-m}_{\mathrm{hol}}(\partial\Omega),\ u \mapsto \Pi Q u$$

with (classical!) pseudodifferential operators $Q \in \Psi^m$ as symbols.

Generalized Toeplitz operators: Properties

(1) For $P \in \Psi^m$ there is a $Q \in \Psi^m$ with $Q\Pi = \Pi Q$ and $T_P = T_Q$ (restriction of Q)

$$\Rightarrow \quad \text{GTO}'\text{s form an algebra}$$

(2) Can define the order $\operatorname{ord}(T_P) \in \mathbb{R}$ and the principal symbol $\sigma(T_P)$ such that

$$\operatorname{ord}(T_{P_1} T_{P_2}) = \operatorname{ord}(T_{P_1}) + \operatorname{ord}(T_{P_2}), \quad \sigma(T_{P_1} T_{P_2}) = \sigma(T_{P_1})(T_{P_2})$$

(3) If $P \in \Psi^m$ and $\sigma(T_P) = 0$, then $\exists Q \in \Psi^{m-1}$ with $T_P = T_Q$.

$$\Rightarrow \quad \operatorname{ord}[T_P, T_Q] \leq \operatorname{ord}(T_P) + \operatorname{ord}(T_Q) - 1$$

(4) If $\operatorname{ord}(T_P) \leq -1$ and $n = \dim(\Omega)$, then $\quad T_P \in \mathcal{S}^p \quad$ for all $p > n$.

$$\Rightarrow \operatorname{ord}[T_{z_j}, T_{z_k}^*] \leq -1 \quad \text{and} \quad [T_{z_j}, T_{z_k}^*] \in \bigcap_{p>n} \mathcal{S}^p(H^2(\partial\Omega))$$

Poisson and trace operators

Let $\Omega \subset \mathbb{C}^n$ (or n-dim. complex mfd) be a smooth strictly pseudoconvex domain.

The Poisson extension operator

$$K: \ C^\infty(\partial\Omega) \to C^\infty(\overline{\Omega}), \qquad \Delta Ku = 0 \text{ on } \Omega, (Ku)|\partial\Omega = u$$

induces invertible bounded operators $W^s_{\text{hol}}(\partial\Omega) \to W^{s+\frac{1}{2}}_{\text{hol}}(\Omega) \quad (s \in \mathbb{R})$.

The boundary value map (trace map)

$$\gamma: \ C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega), \gamma f = f|_{\partial\Omega}$$

induces the inverses $W^{s+\frac{1}{2}}_{\text{hol}}(\Omega) \to W^s_{\text{hol}}(\partial\Omega)$.

Restrictions of analytic functions to submanifolds

Let V be an n-dim. complex submfd. of $U \supset \overline{\mathbb{B}}_d$ open such that

$$V \text{ intersects } \partial\mathbb{B}_d \text{ transversally.}$$

Then $\Omega = \mathbb{B}_d \cap V \subset V$ is strictly pseudoconvex with smooth bdy $\partial\Omega = (\partial\mathbb{B}_d) \cap V$.

Beatrous '86: Restriction defines a surjective bounded operator (k = d - n)

$$R: \; A^2_\alpha = W_{\mathrm{hol}}^{-\frac{\alpha}{2}}(\mathbb{B}_d) \twoheadrightarrow W_{\mathrm{hol}}^{-\frac{\alpha+k}{2}}(\mathbb{B}_d \cap V), f \mapsto f|_{\mathbb{B}_d \cap V}$$

Can show: $\exists$ invertible GTO T_X of order $-\frac{\alpha+k+1}{2}$ such that

$$T: \; A^2_\alpha \overset{R}{\twoheadrightarrow} W_{\mathrm{hol}}^{-\frac{\alpha+k}{2}}(\Omega) \overset{\gamma}{\leftrightarrow} W_{\mathrm{hol}}^{-\frac{\alpha+k+1}{2}}(\partial\Omega) \overset{T_X}{\longleftrightarrow} H^2(\partial\Omega)$$

is onto with $M = \mathrm{Ker}\, T$ and TT^* is a GTO of order 0 on $\partial\Omega$.

Smooth transversal submanifolds

$$T: A^2_\alpha \overset{R}{\twoheadrightarrow} W_{\text{hol}}^{-\frac{\alpha+k}{2}}(\Omega) \overset{\gamma}{\longleftrightarrow} W_{\text{hol}}^{-\frac{\alpha+k+1}{2}}(\partial\Omega) \overset{T_X}{\longleftrightarrow} H^2(\partial\Omega) \text{ onto}$$

Polar decomposition: $T^* = U(TT^*)^{\frac{1}{2}}, \quad U \in L(H^2(\partial\Omega), M^\perp)$ unitary

$$\Rightarrow U = T^*(TT^*)^{-\frac{1}{2}}, \quad P_{M^\perp} = T^*(TT^*)^{-1}T.$$

$$TM_{z_j} = T_X M_{z_j}\gamma R = T_X M_{z_j} T_X^{-1} T \in M_{z_j} T + \mathcal{S}^p \quad \forall p > n$$

$$\Rightarrow S_j = T^*(TT^*)^{-1} T M_{z_j}|_{M^\perp} \overset{p}{\sim} U(TT^*)^{-\frac{1}{2}} M_{z_j} T$$

$$\overset{p}{\sim} UM_{z_j}(TT^*)^{-\frac{1}{2}} T = UM_{z_j}U^* \quad \text{for all } p > n,$$

where we have used that $\operatorname{ord}([T_X, M_{z_j}]T_X^{-1}) = -1 = \operatorname{ord}([(TT^*)^{-\frac{1}{2}}, M_{z_j}])$.

Singularity at the origin

Replace $V \subset U(\overline{\mathbb{B}}_d)$ by

$$\mathcal{L}_V = \bigcup_{z \in V \setminus \{0\}} \{(\mathbb{C}z, cz); c \in \mathbb{C}\} \subset \mathcal{L}_{\mathbb{P}^{d-1}} \text{ tautological line bundle over } \mathbb{P}^{d-1}$$

and $\Omega = V \cap \mathbb{B}_d \subset V$ by the unit disc bundle

$$\Omega = \bigcup_{z \in V \setminus \{0\}} \{(\mathbb{C}z, cz); c \in \mathbb{C} : |cz| < 1\} \subset \mathcal{L}_V.$$

Needs a replacement for the surjectivity result of Beatrous.

Reduction to the pure dimensional case by using the decomposition

$$V = \bigcup_{i=1}^{r} V_i \qquad \text{into irreducible components}$$

References

More details in:

M. Englis, J. Eschmeier, Geometric Arveson-Douglas conjecture, arXiv: 1312.6777

Recent related paper:

R. Douglas, X. Tang, G. Yu: An analytic Riemann Roch theorem, arXiv: 1404.4396