Dissipative and accretive parts of an operator

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Dissipative Operators

Definition

An operator A is *dissipative* if

$$\|(\lambda - A)x\| \geq \lambda\|x\|$$

for all $x \in D(A)$ and $\lambda > 0$.

C_0-semigroups

Definition

A C_0-semigroup T on a Banach space X is a strongly continuous function $T\colon \mathbf{R}_+ \to \mathcal{B}(X)$ such that

$$\begin{aligned} T(0) &= 1 \\ T(s+t) &= T(s)T(t) \qquad (s,t \geq 0). \end{aligned}$$

Its generator A is the operator defined by

$$\begin{aligned} D(A) &= \{x \in X\colon \lim_{t\to 0} \frac{1}{t}(T(t)x - x) \text{ exists}\} \\ Ax &= \lim_{t\to 0} \frac{1}{t}(T(t)x - x). \end{aligned}$$

Lumer-Phillips theorem

Definition

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Theorem

Let A be a densely defined operator. Then A is the generator of a C_0-semigroup of contractions iff

- *A is dissipative*
- *$(\lambda - A)D(A) = X$ for some/all $\lambda > 0$.*

Hille-Yosida theorem

Theorem

Let A be densely defined. Then A generates a C_0-semigroup T with $\|T(t)\| \leq M$ for all $t > 0$ iff

$$\|\lambda^n R(\lambda, A)^n\| \leq M$$

for all $\lambda > 0, n \geq 0$.

Maximal dissipative seminorm

Define the seminorm

$$\begin{aligned} s(x) := \sup\{\, q(x) \colon\ & q \text{ is a seminorm}, \\ & q(y) \le \|y\| \ \forall y \in X, \\ & q((\lambda - A)y) \ge \lambda q(y) \ \forall \lambda > 0, y \in X \,\}. \end{aligned}$$

We call s the maximal seminorm for which A is dissipative.

Example

On a measure space (Ω, μ), let $h\colon \Omega \to \mathbf{C}$ be a measurable function and $A = M_h$ on $L_p(\Omega, \mu)$. Then

$$s(f) = \|f \cdot \mathbf{1}_D\|$$

where $D = \{\, x \in \Omega \colon \operatorname{Re} h(x) \le 0 \,\}$ if $1 \le p < \infty$.

We want to find an alternative expression for the seminorm s. Let A be an operator with non-empty resolvent set $\rho(A)$. Define

$$|||x|||_{A,\mu} = \inf\{ \sum_{k=0}^{n} \|x_i\| \colon x_i \in X, n \geq 0, P(x, x_i, \mu, n) \text{ holds} \}$$

and we say that $P(x, x_i, \mu, n)$ holds if

$$\mu^{-n}(\mu - A)^n(\alpha - A)^{-n}x = \sum_{i=0}^{n} \mu^{-i}(\mu - A)^i(\alpha - A)^{-n}x_i$$

for some $\alpha \in \rho(A)$.

Properties of $|||\cdot|||_{A,\mu}$

Lemma

Let $\rho(A) \neq \emptyset$ *and* $0 < \lambda \leq \mu$. *Then*

- $|||\cdot|||_{A,\mu}$ *is a seminorm,*
- $|||\cdot|||_{A,\mu} \leq \|\cdot\|$,
- $|||\lambda^{-1}(\lambda - A)x|||_{A,\mu} \geq |||x|||_{A,\mu}$ *for all* $x \in D(A)$,
- $|||x|||_{A,\mu} \leq |||x|||_{A,\lambda}$ *for all* $x \in X$.

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- $|||\lambda^{-1}(\lambda - A)x|||_{A,\mu} \geq |||x|||_{A,\mu}$ *for all $x \in D(A)$,*
- $|||x|||_{A,\mu} \leq |||x|||_{A,\lambda}$ *for all $x \in X$.*

So we can define a seminorm $|||\cdot|||_A = \inf_{\mu>0} |||x|||_{A,\mu}$ which satisfies

$$|||(\lambda - A)x|||_A \geq \lambda |||x|||_A$$

for all $\lambda > 0$ and $x \in D(A)$, and it is bounded by the norm $\|\cdot\|$.

Maximal dissipative part

Assume A generates a C_0-semigroup. Let

$$N = \{ x \in X \colon |||x|||_A = 0 \}.$$

Then $R(\alpha, \tilde{A})(x + N) := R(\alpha, A)x + N$ is the resolvent of an operator $\tilde{A}$ on the $|||\cdot|||_A$-completion of

$$X \,/\, N.$$

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We will call this the maximal dissipative part of A. Clearly, $|||\cdot|||_A \leq s(\cdot)$. In fact, $|||\cdot|||_A = s(\cdot)$.

Maximal dissipative part 2

Assume that

$$\mu^{-n}(\mu - A)^n(\alpha - A)^{-n}x = \sum_{i=0}^{n} \mu^{-i}(\mu - A)^i(\alpha - A)^{-n}x_i$$

holds for $\alpha \in \rho(A)$. Then

$$s((\alpha - A)^{-n}x) \leq \sum_{i=0}^{n} \|(\alpha - A)^{-n}x_i\|.$$

Multiply this by α^n and take the limit as $\alpha \in \mathbf{R}, \alpha \to \infty$ to find

$$s(x) \leq \sum_{i=0}^{n} \|x_i\|.$$

So $|||\cdot|||_A \geq s(x)$.

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To take this limit in α, we need that A generates a C_0-semigroup.

Accretive operators

Definition

An operator A is accretive if $-A$ is dissipative, so if

$$\|(\lambda - A)x\| \geq -\lambda\|x\|$$

for all $\lambda < 0$.

So $|||\cdot|||_{-A}$ is an accretive seminorm for A. It is maximal if A generates a C_0-semigroup.

Examples

- Let A be the generator of a hyperbolic semigroup T. Then $X = X_s \oplus X_u$ splits into a stable and an unstable part. The seminorm $|||\cdot|||_A$ vanishes on X_u and is equivalent to $\|\cdot\|$ on X_s. Similarly, $|||\cdot|||_{-A}$ vanishes on X_s and is equivalent to $\|\cdot\|$ on X_u.

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- Let A generate an isometric semigroup. Then $|||\cdot|||_A = |||\cdot|||_{-A} = \|\cdot\|$.

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- Let A generate an isometric semigroup. Then $|||\cdot|||_A = |||\cdot|||_{-A} = \|\cdot\|$.
- Let T be the right shift on $L_p(\mathbf{R}, w(t)\,\mathrm{d}t)$ where

$$w(t) = \begin{cases} e^t & t < 0 \\ e^{-t} & t \leq 0 \end{cases}.$$

Then $|||\cdot|||_A = |||\cdot|||_{-A} = 0$.

Expansive semigroups

Theorem (Goldberg, Smith (1978))

Assume A generates a C_0-semigroup T. The following are equivalent.

- $\|(\lambda - A)x\| \geq -\lambda\|x\|$ *for all* $x \in D(A), \lambda < 0$.
- $\|T(t)x\| \geq \|x\|$ *for all* $x \in X, t > 0$.

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Compare this with the Hille-Yosida theorem. It gives the equivalence of $\|\lambda(\lambda - A)^{-1}\| \leq 1$ for $\lambda > 0$ and $\|T(t)\| \leq 1$ for $t > 0$. In its more general form, the equivalence is between $\|\lambda^n(\lambda - A)^{-n}\| \leq M$ for $\lambda > 0, n \geq 1$ and $\|T(t)\| \leq M$ for $t > 0$.

Generalisation?

So we are wondering whether

$$\|(\lambda - A)^n x\| \geq M(-\lambda)^n \|x\| \qquad (x \in X, \lambda < 0).$$

and

$$\|T(t)x\| \geq \frac{1}{M}\|x\| \qquad (x \in X, t > 0)$$

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are equivalent. We cannot say anything about this, but instead try something different.

The following is done in the proof of the general Hille-Yosida theorem. Assuming that $\|\lambda^n(\lambda - A)^{-n}\| \leq M$ $(\lambda > 0, n \geq 1)$, an equivalent norm $\|\cdot\|'$ is constructed in which $\|\lambda(\lambda - A)^{-1}\|' \leq 1$ $(\lambda > 0)$, so A becomes dissipative in the norm $\|\cdot\|'$. Then, the contractive version applies.

The following is done in the proof of the general Hille-Yosida theorem. Assuming that $\|\lambda^n(\lambda - A)^{-n}\| \leq M$ $(\lambda > 0, n \geq 1)$, an equivalent norm $\|\cdot\|'$ is constructed in which $\|\lambda(\lambda - A)^{-1}\|' \leq 1$ $(\lambda > 0)$, so A becomes dissipative in the norm $\|\cdot\|'$. Then, the contractive version applies.
We can do something similar and find conditions under which the seminorm $|||\cdot|||_{-A}$, which makes A accretive, is a norm and equivalent to $\|\cdot\|$. This is the case, for example, when

$$\|x\| \leq \sum_{i=0}^{n} \|x_i\|$$

whenever $P(x, x_i, \mu, n)$ holds for some $\mu < 0, n \geq 0, x_i \in X$. In that case, there is $c > 0$ such that $\|T(t)x\| \geq c\|x\|$ for all $x \in X, t > 0$. However, the reverse might not hold.

Expansive semigroups 2

Theorem ([Batty & Yeates, 2001])

Let T be a C_0-semigroup on X such that $\|T(t)x\| \geq \|x\|$ for all $x \in X, t > 0$. Then there is a C_0-group S on $Y \supset X$ with

$$S(t)\restriction_X = T(t),$$
$$\|S(t)\| = \|T(t)\|,$$
$$\|S(-t)\| \leq 1$$

for $t > 0$.

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$$\begin{aligned} S(t)\restriction_X &= T(t), \\ \|S(t)\| &= \|T(t)\|, \\ \|S(-t)\| &\leq 1 \end{aligned}$$

for $t > 0$.

The characterisation of the semigroups T that are restrictions of groups S with $\|S(-t)\| \leq M$ for all $t > 0$ is included in [Badea & Müller, 2005].

A corollary

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Corollary

Let A be the generator of a C_0-semigroup T on a Banach space X. The following are equivalent.

- *A is accretive.*
- *There is a C_0-group S on $Y \supset X$ extending T, and S satisfies $\|S(-t)\| \leq 1$ for all $t > 0$.*

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Let us find more general extensions.

Restrictions of C_0-groups

Theorem

Let A be the generator of a C_0-semigroup T on a Banach space X. The following are equivalent.

- *When $P(x, x_i, \mu, n)$ holds for some $x_i \in X, \mu < 0, n \geq 0$ then*

$$\|x\| \leq \sum_{i=0}^{n} \|x_i\|.$$

- *There is a C_0-group S on $Y \supset X$ extending T, and S satisfies $\|S(-t)\| \leq M$ for all $t > 0$.*

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