Generalized Bicircular Projections

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International Workshop on Operator Theory 2014
Belfast, 3–5 September 2014

Definition of a Bicircular Projection

Definition

Let $\mathcal{X}$ be a complex Banach space and let $P : \mathcal{X} \to \mathcal{X}$ be a linear projection. A projection P is called **bicircular** if the mapping $P + \lambda\overline{P}$ is an isometry for all modulus one complex numbers λ.

Example

Every orthogonal projection on a complex Hilbert space is bicircular.

Bicircular Projections and Bicontractive Projections

Every bicircular projection is bicontractive:

$$\|x\| = \|(P - \overline{P})(x)\| = \|2P(x) - x\| \geq 2\|P(x)\| - \|x\|$$

$$\Rightarrow \quad \|P(x)\| \leq \|x\|.$$

Bicircular Projections and Bicontractive Projections

Example

Let $M_2(\mathbb{C})$ be equipped with the spectral norm and let $P : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be defined by

$$P \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}.$$

Then P is a projection such that $\|P\| = \|\overline{P}\| = 1$. However, for $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ we have $\|x\| = 2$, but $\|(P + i\overline{P})(x)\| = \sqrt{2}$. Hence $P + i\overline{P}$ is not an isometry and P is not a bicircular projection.

Symmetric and Antisymmetric Operators

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. Throughout we fix an orthonormal basis $\{e_\lambda : \lambda \in \Lambda\}$ of $\mathcal{H}$.

Let $T \in B(\mathcal{H})$. If $S \in B(\mathcal{H})$ is such that

$$\langle Te_\lambda, e_\mu \rangle = \langle Se_\mu, e_\lambda \rangle \qquad (\lambda, \mu \in \Lambda),$$

then S is called **the transpose of T associated to the basis** $\{e_\lambda : \lambda \in \Lambda\}$ and it is denoted by T^t.

- $S(\mathcal{H}) = \{T \in B(\mathcal{H}) : T^t = T\}$ **symmetric operators**
- $A(\mathcal{H}) = \{T \in B(\mathcal{H}) : T^t = -T\}$ **antisymmetric operators**

Bicircular Projections on Some Operator Spaces

Theorem (L.L. Stachó and B. Zalar, LAA, 2004)

(i) *Let* $P : B(\mathcal{H}) \to B(\mathcal{H})$ *be a bicircular projection. Then* P *has the form* $X \mapsto QX$ *or* $X \mapsto XQ$ *for some* $Q = Q^* = Q^2 \in B(\mathcal{H})$.

(ii) *Let* $P : S(\mathcal{H}) \to S(\mathcal{H})$ *be a bicircular projection. Then either* $P = 0$ *or* $P = I$.

(iii) *Let* $P : A(\mathcal{H}) \to A(\mathcal{H})$ *be a bicircular projection. Then* P *or* $\overline{P}$ *has the form* $X \mapsto QX + XQ^t$ *with* $Q = x \otimes x$ *for some unit vector* $x \in \mathcal{H}$.

Bicircular Projections on C*-algebras

Let A be a C*-algebra.

Every surjective linear isometry $\varphi : A \longrightarrow A$ satisfies

$$\varphi(xy^*x) = \varphi(x)\varphi(y)^*\varphi(x) \qquad (x, y \in A).$$

Every bicircular projection $P : A \to A$ satisfies the functional identity

$$(FI) \qquad P(xyx) = P(x)yx - xP(y^*)^*x + xyP(x) \qquad (x, y \in A).$$

Theorem (M. Fošner and D. I., Comm. Algebra, 2005)

Let R be a 2-torsion free semiprime $*$-ring. Let $P : R \longrightarrow R$ be a projection satisfying the functional identity (FI). Then there exist a $*$-ideal I of R and a selfadjoint projection $p \in M(I^\perp \oplus I^{\perp\perp})$ such that $P(x) = px$ for every $x \in I^\perp$ and $P(x) = xp$ for every $x \in I^{\perp\perp}$.

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Corollary

Let A be a commutative C-algebra and let $P : A \to A$ be a bicircular projection. Then either $P = 0$ or $P = I$.*

Bicircular Projections on C*-algebras

Corollary

Let A be a prime C-algebra and let $P : A \longrightarrow A$ be a bicircular projection. Then there exists $p = p^* = p^2 \in M(A)$ such that P has the form $x \mapsto px$ or $x \mapsto xp$.*

Corollary

Let A be $K(\mathcal{H})$ or $B(\mathcal{H})$ and let $P : A \longrightarrow A$ be a bicircular projection. Then there exists $p = p^* = p^2 \in B(\mathcal{H})$ such that P has the form $x \mapsto px$ or $x \mapsto xp$.

Hermitian Projections

Definition

A bounded linear operator $T : \mathcal{X} \to \mathcal{X}$ is said to be **hermitian** if $e^{i\varphi T}$ is an isometry for all $\varphi \in \mathbb{R}$.

Theorem (J. Jamison, LAA, 2007)

A linear projection on $\mathcal{X}$ is a bicircular projection if and only if it is a hermitian projection.

Definition of a Generalized Bicircular Projection

Definition

A projection $P : \mathcal{X} \to \mathcal{X}$ is called **generalized bicircular** if the mapping $P + \lambda \overline{P}$ is an isometry for some modulus one complex number $\lambda \neq 1$.

- These mappings were first studied by M. Fošner, D. I. and C.K. Li, LAA, 2007.
- The term "generalized bicircular projection"(GBP) first appeared in a paper by F. Botelho and J. Jamison, PAMS, 2008.

GBP on $S_n(\mathbb{C})$

Let A be $S_n(\mathbb{C})$ or $K_n(\mathbb{C})$. A norm $\|\cdot\|$ on A is said to be a **unitary congruence invariant norm** if

$$\|UXU^t\| = \|X\|$$

for all unitary $U \in M_n(\mathbb{C})$ and all $X \in A$.

Theorem (M. Fošner, D. I. and C.K. Li, LAA, 2007)

Let $\|\cdot\|$ be a unitary congruence invariant norm on $S_n(\mathbb{C})$, which is not a multiple of the Frobenius norm, and let $\mathcal{K}$ be the isometry group of $\|\cdot\|$. Suppose $P : S_n(\mathbb{C}) \to S_n(\mathbb{C})$ is a non-trivial linear projection and $\lambda \neq 1$ a modulus one complex number. Then $P + \lambda\overline{P} \in \mathcal{K}$ if and only if $\lambda = -1$ and there exists $Q = Q^* = Q^2 \in M_n(\mathbb{C})$ such that P or $\overline{P}$ has the form $X \mapsto QXQ^t + (I - Q)X(I - Q^t)$.

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GBP on $K_n(\mathbb{C})$

Theorem (M. Fošner, D. I. and C.K. Li, LAA, 2007)

Let $n \geq 3$ and $\|\cdot\|$ be a unitary congruence invariant norm on $K_n(\mathbb{C})$, which is not a multiple of the Frobenius norm. Let $\mathcal{K}$ be the isometry group of $\|\cdot\|$. Suppose $P : K_n(\mathbb{C}) \to K_n(\mathbb{C})$ is a non-trivial linear projection and $\lambda \neq 1$ a modulus one complex number. Then $P + \lambda\overline{P} \in \mathcal{K}$ if and only if one of the following holds.

(a) *There exists $Q = vv^*$ for a unit vector $v \in \mathbb{C}^n$ such that P or $\overline{P}$ has the form $X \mapsto QX + XQ^t$.*

(b) *$\lambda = -1$, $\mathcal{K} = G$ and there exists $Q = Q^* = Q^2 \in M_n(\mathbb{C})$ such that P or $\overline{P}$ has the form $X \mapsto QXQ^t + (I - Q)X(I - Q^t)$.*

(c) *$(\lambda, n) = (-1, 4)$, $\psi \in \mathcal{K}$, and there is $U \in U(\mathbb{C}^4)$, satisfying $\psi(U^tXU) = \overline{U}\psi(X)U^*$ for all $X \in K_4(\mathbb{C})$, such that P or $\overline{P}$ has the form $X \mapsto (X + \psi(U^tXU))/2 = (X + \overline{U}\psi(X)U^*)/2$.*

GBP on Arbitrary Complex Banach Spaces

Theorem (P.-K. Lin, JMAA, 2008)

Every generalized bicircular projection is bicontractive.

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Let $\mathcal{X}$ be a complex Banach space and let $P : \mathcal{X} \to \mathcal{X}$ be a linear projection. Then $P + \lambda\overline{P}$ is an isometry for some modulus one complex number $\lambda \neq 1$ if and only if one of the following holds:

(i) *P is hermitian ($\equiv$ bicircular),*

(ii) *$\lambda = e^{\frac{2\pi i}{n}}$ for some integer $n \geq 2$.*

Furthermore, if n is any integer such that $n \geq 2$, then for $\lambda = e^{\frac{2\pi i}{n}}$ there is a complex Banach space $\mathcal{X}$ and a nontrivial linear projection P on $\mathcal{X}$ such that $P + \lambda\overline{P}$ is an isometry.

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JB*-triples

A **JB*-triple** is a complex Banach space A together with a continuous triple product $\{\cdot\cdot\cdot\} : A \times A \times A \to A$ such that

(i) $\{xyz\}$ is linear in x and z and conjugate linear in y;

(ii) $\{xyz\} = \{zyx\}$;

(iii) for any $x \in A$, the operator $\delta(x) : A \to A$ defined by $\delta(x)y = \{xxy\}$ is hermitian with nonnegative spectrum;

(iv) $\delta(x)\{abc\} = \{\delta(x)a, b, c\} - \{a, \delta(x)b, c\} + \{a, b, \delta(x)c\}$;

(v) for every $x \in A$, $\|\{xxx\}\| = \|x\|^3$.

- complex Hilbert spaces: $\{xyz\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$
- C*-algebras, $S(\mathcal{H})$, $A(\mathcal{H})$: $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$.

GBP on JB*-triples

Theorem (D. I., LAA, 2010)

Let A be a JB-triple and let $P : A \longrightarrow A$ be a rank one linear projection. Then P is bicontractive if and only if P is hermitian ($\equiv$ bicircular).*

Theorem (D. I., LAA, 2010)

Let A be a JB-triple and let $P : A \longrightarrow A$ be a linear projection. Then $P + \lambda\overline{P}$ is an isometry for some modulus one complex number $\lambda \neq 1$ if and only if one of the following holds:*

(i) *P is hermitian ($\equiv$ bicircular),*

(ii) *$\lambda = -1$ and $P = \frac{1}{2}(I + \varphi)$ for some linear isometry $\varphi : A \longrightarrow A$ satisfying $\varphi^2 = I$.*

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(ii) *$\lambda = -1$ and $P = \frac{1}{2}(I + \varphi)$ for some linear isometry $\varphi : A \longrightarrow A$ satisfying $\varphi^2 = I$.*

Applications to Some Important JB*-triples

$A = B(\mathcal{H})$ or $A = K(\mathcal{H})$:

(i) P has the form $X \mapsto QX$ or $X \mapsto XQ$ for some $Q = Q^* = Q^2 \in B(\mathcal{H})$,

(ii) $\lambda = -1$ and P has one of the following forms:

- $X \mapsto \frac{1}{2}(X + UXV)$ for unitary $U, V \in B(\mathcal{H})$ such that $U^2 = \mu I$, $V^2 = \overline{\mu} I$ for some $\mu \in \mathbb{C}$, $|\mu| = 1$,
- $X \mapsto \frac{1}{2}(X + UX^t V)$ for unitary $U, V \in B(\mathcal{H})$ such that $V = \pm(U^t)^*$.

Applications to Some Important JB*-triples

$A = C_0(\Omega)$:

(i) $P = 0$ or $P = I$,

(ii) $\lambda = -1$ and there exist a homeomorphism $\varphi : \Omega \to \Omega$ satisfying $\varphi^2 = I$ and a continuous function $u : \Omega \to \mathbb{C}$ satisfying $|u(w)| = 1$ and $u(\varphi(w)) = \overline{u(w)}$ for every $w \in \Omega$, such that

$$P(f)(w) = \frac{1}{2}\Big(f(w) + u(w)f\big(\varphi(w)\big)\Big)$$

for all $f \in C_0(\Omega)$, $w \in \Omega$.

GBP on Some Matrix and Operator Spaces

- bicircular projections on $S(\mathcal{H})$ and $A(\mathcal{H})$:
  L.L. Stachó and B. Zalar, LAA, 2004
- generalized bicircular projections on $S_n(\mathbb{C})$ and $K_n(\mathbb{C})$:
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Surjective Linear Isometries on $S(\mathcal{H})$ and $A(\mathcal{H})$

Every surjective linear isometry $\varphi : A \to A$, where A is $\mathcal{S}(\mathcal{H})$ or $A(\mathcal{H})$, satisfies

$$\varphi(XY^*X) = \varphi(X)\varphi(Y)^*\varphi(X)$$

for all $X, Y \in A$.

The following theorem gives an explicit formula for φ.

Theorem (A. Fošner and D. I., OaM, 2011)

Let A be $S(\mathcal{H})$ or $A(\mathcal{H})$ and let $\varphi : A \to A$ be a surjective linear isometry. Then there exists a unitary $U \in B(\mathcal{H})$ such that φ has the form $X \mapsto UXU^t$.

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GBP on $S(\mathcal{H})$ and $A(\mathcal{H})$

Corollary

Let $P : S(\mathcal{H}) \to S(\mathcal{H})$ be a nontrivial linear projection and $\lambda \neq 1$ a modulus one complex number. Then $P + \lambda \overline{P}$ is an isometry if and only if $\lambda = -1$ and there exists $Q = Q^* = Q^2 \in B(\mathcal{H})$ such that P or $\overline{P}$ has the form $X \mapsto QXQ^t + (I - Q)X(I - Q^t)$.

Corollary

Let $P : A(\mathcal{H}) \to A(\mathcal{H})$ be a nontrivial linear projection and $\lambda \neq 1$ a modulus one complex number. Then $P + \lambda \overline{P}$ is an isometry if and only if one of the following holds:

- **(i)** *P or $\overline{P}$ has the form $X \mapsto QX + XQ^t$, where $Q = x \otimes x$ for some norm one $x \in \mathcal{H}$,*
- **(ii)** *$\lambda = -1$ and there exists $Q = Q^* = Q^2 \in B(\mathcal{H})$ such that P or $\overline{P}$ has the form $X \mapsto QXQ^t + (I - Q)X(I - Q^t)$.*

Basic Problem

Problem

Determine the structure of generalized bicircular projections on a given complex Banach space
(decompose I as a sum of linear projections P_1 and P_2 such that $P_1 + \lambda P_2$ is an isometry for some modulus one complex number $\lambda \neq 1$).

Example

$P = \frac{1}{2}(I + \varphi)$ for some linear isometry φ satisfying $\varphi^2 = I$.

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Related Problems

Problem

Characterization of projections that are the average (or more generally, the convex combination) of two (or more) surjective isometries.

Problem

Decomposition of I *as a sum of linear projections* P_1, P_2 *and* P_3 *such that* $P_1 + \lambda P_2 + \mu P_3$ *is an isometry for some modulus one complex numbers* λ *and* μ.

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