BUAP

The inverse along an operator

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Motivation

Let X be a Banach space, and let $B(X)$ denote the set of bounded linear operators on X.

Problem

Given $A \in B(X)$ and $y \in X$, find $x \in X$ such that

$$Ax = y.$$

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If A is invertible, then we may take

$$x = A^{-1}y.$$

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Now suppose A is not invertible and

- There exist a closed subspace N such that

$$X = N \oplus \mathcal{N}(A).$$

- $\mathcal{R}(A)$ is closed and there exist a closed subspace M such that

$$X = \mathcal{R}(A) \oplus M.$$

Motivation

The operator $A : X \to X$ has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} N \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ M \end{bmatrix}$$

where $A_1 : N \to \mathcal{R}(A)$ is 1-1 and onto and $A_2 : \mathcal{N}(A) \to M$ arbitrary.

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Let us consider the operator $B \in B(X)$ defined by

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From $Ax = y$ we have that $y \in \mathcal{R}(A) = \mathcal{R}(AB)$, so $ABy = y$.

Taking $x = (I - BA)z + By$ we get

$$Ax = (A - ABA)z + ABy = y.$$

Generalized inverses

$B(X)$ the algebra of bounded linear operator.
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  and we write $B = A^{(2)}$.

Outer inverses

If B is an outer inverse for A, then from

$$B = BAB$$

we have

- BA is a projection on $R(B)$.
- $I - AB$ is a projection on $N(B)$.

Let $T, S \subset X$, if $R(B) = T$ and $N(B) = S$ then we write

$$B = A^{(2)}_{T,S}.$$

Matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} T \\ N(BA) \end{bmatrix} \to \begin{bmatrix} A(T) \\ S \end{bmatrix}$$

$$A^{(2)}_{T,S} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \to \begin{bmatrix} T \\ N(BA) \end{bmatrix}$$

Prescribed idempotents

Djordjevic and Wei (2005)

Let $a \in \mathcal{R}$ and let $p, q \in \mathcal{R}$ be idempotents, an element $b \in \mathcal{R}$ satisfying

$$bab = b, \quad ba = p \quad \text{and} \quad 1 - ab = q,$$

will be called a (p, q)-inverse of a, and we write $b = a^{(2)}_{p,q}$.

Principal ideals

For an element $a \in \mathcal{R}$, we define the following image ideals

$$a\mathcal{R} := \{ax : x \in \mathcal{R}\}, \qquad \mathcal{R}a := \{xa : x \in \mathcal{R}\},$$

and kernel ideals

$$a^0 := \{x \in \mathcal{R} : ax = 0\}, \qquad {}^0a =: \{x \in \mathcal{R} : xa = 0\}.$$

Prescribed ideals

Kantun-Montiel (2013)

Let $p, q \in \mathcal{R}$ be idempotents and $a \in \mathcal{R}$. An element $b \in \mathcal{R}$ is an image-kernel (p, q)-inverse of a if

$$bab = b, \quad ba\mathcal{R} = p\mathcal{R} \quad \text{and} \quad (1 - ab)\mathcal{R} = q\mathcal{R}.$$

The following are equivalent:

1. $ba\mathcal{R} = p\mathcal{R}$;
2. $\mathcal{R}(1 - ba) = \mathcal{R}(1 - p)$;
3. ${}^0(ba) = {}^0p$;
4. $(1 - ba)^0 = (1 - p)^0$.

The following are equivalent:

1. $(1 - ab)\mathcal{R} = q\mathcal{R}$;
2. $\mathcal{R}ab = \mathcal{R}(1 - q)$;
3. ${}^0(1 - ab) = {}^0q$;
4. $(ab)^0 = (1 - q)^0$.

Inverse along an element

X. Mary (2011)

An element $a \in \mathcal{R}$ is invertible along $d \in \mathcal{R}$ if there exists $b \in \mathcal{R}$ such that

$$bab = b, \quad b\mathcal{R} = d\mathcal{R}, \quad \mathcal{R}b = \mathcal{R}d \tag{1}$$

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Let S be a semigroup and $a \in S$.

1. a is group invertible if and only if it is invertible along a. In this case the inverse along a is inner and coincides with the group inverse.
2. a is Drazin invertible if and only if it is invertible along some a^k, $k \in \mathbb{N}$, and in this case the two inverses coincide.
3. If S is a *-semigroup, a is Moore-Penrose invertible if and only if it is invertible along a^*. In this case the inverse along a^* is inner and coincides with the Moore-Penrose inverse.

Inverse along an element

Wei (1998)

Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m-s$. In addition, suppose $G \in \mathbb{C}^{n\times m}$ such that $R(G) = T$ and $N(G) = S$. If A has an outer inverse $A^{(2)}_{T,S}$ then AG and GA are group invertible. Further, we have

$$A^{(2)}_{T,S} = G(AG)^{\sharp} = (GA)^{\sharp}G.$$

Mary (2011)

Let $a, d \in \mathcal{R}$. The following statements are equivalent:

- a is invertible along d;
- $\mathcal{R}d = \mathcal{R}ad$ and $(ad)^{\sharp}$ exists;
- $d\mathcal{R} = da\mathcal{R}$ and $(da)^{\sharp}$ exists;

In this case, $a^{\|d} = d(ad)^{\sharp} = (da)^{\sharp}d$.

Inverse along an operator

Kantun-Montiel (2013)

Let $A, T \in B(X)$ be nonzero operators. The following statements are equivalent:

1. B is the inverse of A along T.
2. B is an outer inverse of A such that $R(B) = R(T)$ and $N(B) = N(T)$.

Invertibility along an operator

Theorem (Kantun-Montiel and Djordjevic)

Let $A, T \in B(X)$ be nonzero operators. The following statements are equivalent.

1. A is invertible along T.
2. $\mathcal{R}(T)$ is closed and complemented subspace of X, $A(\mathcal{R}(T)) = \mathcal{R}(AT)$ is closed such that $\mathcal{R}(AT) \oplus \mathcal{N}(T) = X$ and the reduction $A|_{\mathcal{R}(T)} : \mathcal{R}(T) \to \mathcal{R}(AT)$ is invertible.

Spectral projections

Theorem (Cho and Kantun-Montiel, 2013)

Let $A \in B(X)$ and Λ be a spectral set for A. If $0 \notin \Lambda$ then A is invertible along $P_\Lambda(A)$.

Corollary

Let $A \in B(X)$ and Λ be a spectral set for A. If $0 \in \Lambda$ then A is invertible along $I - P_\Lambda(A)$.

More Drazin inverses

Drazin (2012)

Let $x, y \in \mathcal{R}$, an element $a \in \mathcal{R}$ is *Drazin* (x, y)*-invertible* if there exists $b \in \mathcal{R}$ such that

$$bab = b, \quad b\mathcal{R} = x\mathcal{R} \quad \text{and} \quad \mathcal{R}b = \mathcal{R}y.$$

Drazin(2012)

Let $a, b, x, y \in \mathcal{R}$. We call b an *annihilator* (x, y)*-inverse* of a if

$$bab = b, \quad {}^0b = {}^0x \quad \text{and} \quad b^0 = y^0.$$

More Drazin inverses

Kantun-Montiel (2013)

Let H be a Hilbert space and $A, B, U, V \in H$. B is an annihilator (U, V)-inverse of A if and only if $R(B) = \overline{R(T)}$ and $N(B) = N(T)$.

Thank you.