Spectral behaviour of contractions

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September 3–5, 2014, Belfast, IWOP

Invariant subspace lattices

$\mathcal{H}$ complex Hilbert space, $\dim \mathcal{H} = \aleph_0$

$\mathcal{L}(\mathcal{H})$ bounded, linear operators on $\mathcal{H}$

$T \in \mathcal{L}(\mathcal{H})$

Lat T : lattice of all invariant subspaces of T

Hlat T : lattice of all hyperinvariant subspaces of T

Question (ISP)

Is Lat T non-trivial for every $T \in \mathcal{L}(\mathcal{H})$?

Question (HSP)

Is Hlat T non-trivial for every $T \in \mathcal{L}(\mathcal{H}) \setminus \mathbb{C}I$?

H. RADJAVI and P. ROSENTHAL: *Invariant subspaces*, 1973, 2003

I. CHALENDAR and J.R. PARTINGTON: *Modern approaches to the invariant-subspace problem*, 2011

Contractions

We may assume that T is an *absolutely continuous* (a.c.) contraction:

$\|T\| \leq 1$,

$T = U \oplus T_c$, where U a.c. unitary and T_c c.n.u. contraction.

We shall assume that the a.c. contraction T is *asymptotically non-vanishing* (a.n.v.):

$$\exists h \in \mathcal{H}, \quad \lim_n \|T^n h\| > 0.$$

Unitary asymptote

Definition

(X, V) is a *unitary asymptote* of T:

$V \in \mathcal{L}(\mathcal{K})$ a.c. unitary operator,

$X \in \mathcal{L}(\mathcal{H}, \mathcal{K}),\ \|Xh\| = \lim_n \|T^n h\|\ \forall h \in \mathcal{H},\ \vee_{n=1}^{\infty} V^{-n} X \mathcal{H} = \mathcal{K}$,

$XT = VX$

$\omega(T)$ measurable support of the spectral measure E of V:

$$E(\alpha) = 0 \iff m(\alpha \cap \omega(T)) = 0 \quad (\alpha \subset \mathbb{T})$$

Definition

$\omega(T)$ is the *residual set* of T

Functional calculus

$\Phi_T : H^\infty \to \mathcal{L}(\mathcal{H})$ *Sz.-Nagy–Foias functional calculus* for T:

contractive, weak-$*$ continuous, algebra-homomorphism,

$1 \mapsto I, \ \chi \mapsto T \quad (\chi(z) = z)$.

$f_2 \prec f_1 \ (|f_2(z)| \leq |f_1(z)| \ \forall z \in \mathbb{D}) \implies f_2(T) \prec f_1(T) \ (\|f_2(T)h\| \leq \|f_1(T)h\| \ \forall h \in \mathcal{H})$

$F = \{f_n\}_{n=1}^\infty \subset H^\infty$ decreasing ($f_{n+1} \prec f_n \ \forall n$):

$\varphi_F(\zeta) = \lim_n |f_n(\zeta)|$ for a.e. $\zeta \in \mathbb{T}$,

$N_F = \{\zeta \in \mathbb{T} : \varphi_F(\zeta) > 0\}$,

$\mathcal{H}_0(T, F) = \{h \in \mathcal{H} : \lim_n \|f_n(T)h\| = 0\} \in \text{Hlat } T$.

$\pi(T)$ largest measurable set on $\mathbb{T}$ such that

$$m(N_F \cap \pi(T)) > 0 \implies \mathcal{H}_0(T, F) = \{0\}$$

Quasianalytic spectral set

Definition

$\pi(T)$ is the *quasianalytic spectral set* of T

Definition

T is a *quasianalytic contraction* if $\pi(T) = \omega(T)$

Proposition

If the a.n.v. contraction T is NOT quasianalytic, then Hlat T is non-trivial.

What is the spectral behaviour of quasianalytic contractions?

Possible constraints provide hyperinvariant subspace theorems.

$\mathcal{L}_{\text{qa}}(\mathcal{H})$ collection of all quasianalytic contractions on $\mathcal{H}$

C_{10}-contractions

Proposition

If $T \in \mathcal{L}_{\mathrm{qa}}(\mathcal{H})$ *then* $T \in C_{10}$*, that is*

$$\lim_n \|T^{*n}h\| = 0 < \lim_n \|T^n h\| \quad \forall 0 \neq h \in \mathcal{H}.$$

Spectral characterization of C_{10}-contractions is known.

$T \in C_{10}$

$\sigma(V)$ is the essential support of $\omega(T)$: $\sigma(V) = \mathrm{es}(\omega(T))$

$\sigma(V)$ is *neatly contained* in $\sigma(T)$:

$\sigma(V) \subset \sigma(T)$,

$m(\sigma(V) \cap \sigma') > 0$ if $\emptyset \neq \sigma' \subset \sigma(T)$ is closed and $\sigma(T) \setminus \sigma'$ is closed

This is the only constraint even in the cyclic case.

Questions and obstacles

Question (1.)

Given $\omega_0 \subset \mathbb{T}$ *of positive measure and compact* $\sigma \subset \mathbb{D}^-$ *such that* $\mathrm{es}(\omega_0)$ *is neatly contained in* σ*,*

$$\exists ?\ T \in \mathcal{L}_{\mathrm{qa}}(\mathcal{H}),\ \sigma(T) = \sigma \ \text{ and } \ \omega(T) = \omega_0 ?$$

Construction of a C_{10}-contraction T satisfying $\omega(T) = \omega_0$ and $\sigma(T) = \mathrm{es}(\omega_0)$:

$T = W|\mathcal{M}$, W bilateral weighted shift, $\mathcal{M} \in \mathrm{Lat}\, W$,

$\sum_{n=1}^{\infty} n^p \|T^{-p}\| < \infty$ with some integer p.

Under these conditions T is necessarily non-quasianalytic.

Question (2.)

Given closed arc $J \subset \mathbb{T}$ *and* $c > 0$*,*

$$\exists ?\ T \in \mathcal{L}_{\mathrm{qa}}(\mathcal{H}),\ \sigma(T) = \pi(T) = J \ \text{ and } \ \|T^{-1}\| > c ?$$

Contractions with full residual set

$\mathrm{Lat_s}\, T$ collection of *shift-type invariant subspaces*:

$$T|\mathcal{M} \text{ is similar to } S \in \mathcal{L}(H^2),\ Sf = \chi f.$$

Proposition

If $\omega(T) = \mathbb{T}$, *then* $\vee \mathrm{Lat_s}\, T = \mathcal{H}$.

Proposition

$\forall T_1 \in \mathcal{L}_{\mathrm{qa}}(\mathcal{H}),\ \exists T_2 \in \mathcal{L}_{\mathrm{qa}}(\mathcal{H}),\ \omega(T_2) = \mathbb{T},\ \{T_2\}' \supset \{T_1\}'$ *and so* $\mathrm{Hlat}\, T_2 \subset \mathrm{Hlat}\, T_1$.

(HSP) for a.n.v. contractions can be reduced to the case when T is quasianalytic and $\pi(T) = \mathbb{T}$. Then $\sigma(T)$ is connected.

Question (3.)

Given connected, compact $\mathbb{T} \subset \sigma \subset \mathbb{D}^-$,

$$\exists?\ T \in \mathcal{L}_{\mathrm{qa}}(\mathcal{H}),\ \sigma(T) = \sigma \text{ and } \pi(T) = \mathbb{T}?$$

Connection between two questions

Theorem

Positive answer for Question 2 implies affirmative answer for Question 3

in the special case when $\sigma = K^2$ *with a connected, compact* $\mathbb{T}_+ \subset K \subset \mathbb{D}_+^-$.

$$\mathbb{D}_+ = \{z \in \mathbb{D} : \operatorname{Im} z > 0\}, \quad \mathbb{T}_+ = \{\zeta \in \mathbb{T} : \operatorname{Im} \zeta \geq 0\}, \quad K^2 = \{z^2 : z \in K\}$$

$\sigma = \mathbb{T} \cup \{\rho(t) e^{i\varphi(t)} : t \in [0, 1)\}$ $\quad (0 \leq \rho(t) \to 1,\ 0 \leq \varphi(t) \to \infty$ increasingly)

spiral is not of the form $\sigma = K^2$

Proof

$\{\lambda_n\}_{n=1}^{\infty}$ dense in K

$\Omega_n = \{z \in \mathbb{C} : \mathrm{dist}(z, K) < 1/n\}$ connected, open

$\lambda'_n \in \mathbb{D} \cap \Omega_n, \ |\lambda_n - \lambda'_n| < 1/(2n)$

$\Gamma_n \subset (\Omega_n \cap \mathbb{D}) \cup \{-1, 1\}$ simple rectifiable curve, with endpoints $-1, 1$

G_n simply connected domain, bounded by $\mathbb{T}_+ \cup \Gamma_n, \ G_n \subset \Omega_n, \ \lambda'_n \in G_n$

$f_n \colon \mathbb{D} \to G_n$ conformal surjection, $f_n(0) = \lambda'_n$

$J_n = f_n^{-1}(\mathbb{T}_+)$ closed arc

$\exists T_n \in \mathcal{L}_{\mathrm{qa}}(\mathcal{H}_n), \ \sigma(T_n) = \pi(T_n) = J_n, \ \|T_n^{-1}\| > n$

$\widetilde{T}_n = f_n(T_n) \in \mathcal{L}_{\mathrm{qa}}(\mathcal{H}_n), \ \sigma(\widetilde{T}_n) = \pi(\widetilde{T}_n) = \mathbb{T}_+$

$e_n \in \mathcal{H}_n, \ \|e_n\| = 1$ and $\|T_n e_n\| < 1/n \implies \|\widetilde{T}_n e_n - \lambda'_n e_n\| \leq 2/n$

$\widetilde{T} = \sum_n \oplus \widetilde{T}_n, \ \sigma(\widetilde{T}) = K$ and $\pi(\widetilde{T}) = \mathbb{T}_+$

$T = \widetilde{T}^2$ quasianalytic, $\sigma(T) = K^2 = \sigma$ and $\pi(T) = \mathbb{T}_+^2 = \mathbb{T}$

Asymptotically cyclic contractions

Definition

T is *asymptotically cyclic*, if V is cyclic

$\{T\}'$ can be identified with a quasianalytic function algebra $H^\infty \subset \mathcal{F}(T) \subset L^\infty(\mathbb{T})$

(ISP) can be reduced to this case

Definition

$\mathcal{L}_0(\mathcal{H}) = \{T \in \mathcal{L}_{\mathrm{qa}}(\mathcal{H}) : T \text{ is asymptotically cyclic}\}$

Definition

$\mathcal{L}_1(\mathcal{H}) = \{T \in \mathcal{L}_0(\mathcal{H}) : \pi(T) = \mathbb{T}\}$

Spectra in the asymptotically cyclic case

Proposition (K, Totik)

$\forall T_0 \in \mathcal{L}_0(\mathcal{H}),\ \exists T_1 \in \mathcal{L}_1(\mathcal{H}),\ \{T_0\}' = \{T_1\}'$ *and so* $\text{Hlat}\, T_0 = \text{Hlat}\, T_1$

Question (4.)

What are the possible spectra of contractions belonging to $\mathcal{L}_1(\mathcal{H})$*?*

Theorem

$\forall c > 1,\ \exists T \in \mathcal{L}_1(\mathcal{H}),\ \sigma(T) = \mathbb{T}$ *and* $\|T^{-1}\| > c$

Proof

$\beta\colon \mathbb{Z} \to [1,\infty),\ \beta(n) = 1\ \forall n \geq 0,\ \beta(-n) = e^{\varphi(n)}\ \forall n \in \mathbb{N}$

$\varphi\colon \mathbb{N} \to [1,\infty)$ increasing, $\lim_n \varphi(n) = \infty$

$L^2(\beta) = \left\{ f \in L^2(\mathbb{T}) : \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \beta(n)^2 < \infty \right\}$ Hilbert space

$T_\beta \in \mathcal{L}(L^2(\beta)),\ T_\beta f = \chi f$ asymptotically cyclic C_{10}-contraction

$\{q_k\}_{k=1}^\infty \subset (0,1)$ decreasing, $\lim_k q_k = 0$
$\{p_k\}_{k=1}^\infty \subset \mathbb{N}$ increasing, $p_1 = 1$ and $\sum_{n=p_k+1}^{p_{k+1}} 1/n \geq 1/q_k\ \forall k \in \mathbb{N}$

$\varphi(1) := c,\ \varphi(n) := \varphi(p_k) + (n - p_k)q_k \quad \forall p_k < n \leq p_{k+1}\ (k \in \mathbb{N})$

Then

$$\sum_{n=1}^{\infty} \frac{\log \beta(-n)}{n^2} = c + \sum_{k=1}^{\infty} \sum_{n=p_k+1}^{p_{k+1}} \frac{\varphi(n)}{n^2} = \infty \implies T_\beta \text{ quasianalytic;}$$

$$r(T_\beta^{-1}) = \lim_n \|T_\beta^{-n}\|^{1/n} = \lim_n e^{\varphi(n)/n} = 1 \implies \sigma(T_\beta) = \mathbb{T}.$$

Example

$\delta \in (0,1), \quad \Omega_\delta = \left\{ z = re^{it} : \delta < r < 1,\ 0 < t < \pi \right\}$

$$\eta_\delta : \mathbb{D} \to \Omega_\delta \ \text{ conformal surjection}, \quad \vartheta_\delta = \eta_\delta^2$$

$T \in \mathcal{L}_1(\mathcal{H})$ with $\sigma(T) = \mathbb{T}$

Then

$$T_\delta = \vartheta_\delta(T) \in \mathcal{L}_1(\mathcal{H}) \quad \text{and} \quad \sigma(T_\delta) = \mathbb{T} \cup \delta\mathbb{T} \cup [\delta, 1]$$

References

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