SCHUR MULTIPLIERS OF CARTAN PAIRS

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PLAN

1. Classical Schur multipliers
2. Cartan Pairs
3. Feldman–Moore relations
4. Schur multipliers revisited
5. Grothendieck multipliers
6. The hyperfinite II_1 factor

① CLASSICAL SCHUR MULTIPLIERS

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$$\underset{\text{linear}}{\Phi : \mathcal{B}(\ell^2) \to \mathcal{B}(\ell^2)} \qquad \Phi(D_1 T D_2) = D_1 \Phi(T) D_2$$

$$(T \in \mathcal{B}(\ell^2),\ D_1, D_2 \in \mathcal{D})$$

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$$\underset{w^*\text{-cts}}{\Longrightarrow} \quad \Phi(T) = \underset{\substack{\uparrow \\ \text{Schur product}}}{\varphi \bullet T} \qquad (T \in \mathcal{B}(\ell^2))$$

$$\varphi \bullet T = [\varphi_{ij} t_{ij}]_{i,j \in \mathbb{N}}$$

Defn $\varphi : \mathbb{N}\times\mathbb{N} \to \mathbb{C}$ is a (classical) Schur multiplier

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__Thm__ $\left\{ \begin{array}{c} w^*\text{-cts } \mathcal{D}\text{-bimodule maps} \\ \mathcal{B}(\ell^2) \longrightarrow \mathcal{B}(\ell^2) \end{array} \right\} = \left\{ \begin{array}{c} T \longmapsto \varphi \bullet T, \\ \varphi \text{ a Schur mult.} \end{array} \right\}$

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Remark $\{\text{Schur multipliers}\} \subsetneq \ell^\infty(\mathbb{N}\times\mathbb{N})$

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<u>Defn</u> $\mathcal{M}(R)$ = vN alg generated by $\{L(a) : \text{supp } a \text{ small}\}$
where $L(a)\xi := a * \xi \quad (\xi \in H)$

$R = (R, X, \mu)$ a Feldman–Moore relation

$M(R) = \mathrm{vN\,alg}\{L(a) : \operatorname{supp} a \text{ small}\} \subseteq B(H)$

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Thm (Feldman–Moore, 1977)

1. $(\mathcal{M}(\mathcal{R}), \mathcal{D}(\mathcal{R}))$ is a Cartan pair
2. Up to isomorphism, every (separable) Cartan pair arises in this way

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<u>Thm</u> (L-Sprunk-Todorov-Turowska)

$\Phi : \mathcal{M}(R) \to \mathcal{M}(R)$ a w^*-cts $\mathcal{D}(R)$ bimodule map $\iff$ $\Phi(L(a)) = L(\varphi \bullet a)$, some $\varphi \in \mathfrak{S}(R)$

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Notation: For $a, b\ X \to \ell^2$, let $\langle a,b\rangle : X \times X \to \mathbb{C}$,

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Thm (Grothendieck)

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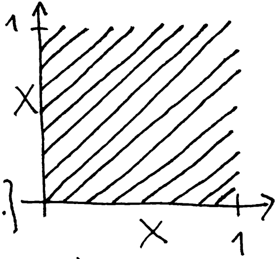

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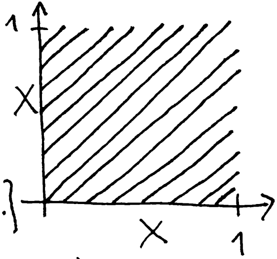

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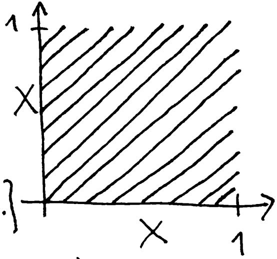

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__Remark__ Pop–Smith ('95) define $\bullet : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$

Our definition extends this to $\mathfrak{S}(\mathcal{R}) \times \mathcal{R} \to \mathcal{R}$.

For $S \subseteq G$, let $\Delta(S)$ be the characteristic function of $\left\{ (x, x+s) : \begin{array}{l} x \in [0,1) \\ s \in S \end{array} \right\}$

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<u>Thm</u> (LSTT)

$\emptyset \neq S \subset G$, $\Delta(S) \in A(R) \Longrightarrow G \setminus S$ not dense in $[0,1)$.

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$\emptyset \neq S \subset G$, $\Delta(S) \in A(R) \implies G \setminus S$ not dense in $[0,1)$.

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Thm $\Rightarrow \Delta(S) \notin \mathbb{A}(R)$. ■