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Levinson's operator inequality and its converses

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History of Levinson's inequality

Norman Levinson

Lynn, Massachusetts (1912) - Boston (1975)

N. Levinson, *Generalization of an inequality of Ky Fan*, J. Math. Anal. Appl. **8** (1964), 133-134.

The following inequality is proven

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The following inequality is proven

Theorem A

If $f : (0, 2c) \to \mathbb{R}$ satisfies $f''' \geq 0$ and p_i, x_i, y_i, $i = 1, 2, \ldots, n$, are such that $p_i > 0$, $\sum_{i=1}^n p_i = 1$, $0 \leq x_i \leq c$ and

$$x_1 + y_1 = x_2 + y_2 = \ldots = x_n + y_n = 2c, \tag{1}$$

then the inequality

$$\sum_{i=1}^n p_i f(x_i) - f(\bar{x}) \leq \sum_{i=1}^n p_i f(y_i) - f(\bar{y}) \tag{2}$$

holds, where $\bar{x} = \sum_{i=1}^n p_i x_i$ and $\bar{y} = \sum_{i=1}^n p_i y_i$.

Proof.

Let a be a constant, $0 < a \leq c$, to be chosen later. Let $F(u) = f(u) - f(2c-u)$. Then for $0 < u \leq c$ and some θ, $0 < \theta < 1$,

$$F(u) = F(a) + (u-a)F'(a) + \frac{1}{2}(u-a)^2 F''(a + \theta(u-a)). \qquad (3)$$

Clearly

$$F''(a+\theta(u-a)) = f''(a+\theta(u-a)) - f''(2c-a-\theta(u-a))$$

or

$$F''(a+\theta(u-a)) = -2[c-a-\theta(u-a)]f'''(u_1)$$

where $a+\theta(u-a) < u_1 < 2c-a-\theta(u-a)$ and the bracket in the above equality is nonnegative since $a \leq c$ and $u \leq c$. Hence since $f''' \geq 0$

$$F''(a+\theta(u-a)) \leq 0.$$

Thus (3) gives

$$f(u) - f(2c-u) \leq f(a) - f(2c-a) + (u-a)F'(a). \qquad (4)$$

In (4) set $u = x_i$, multiply by p_i and sum. Set $a = \sum_{i=1}^{n} p_i x_i$. This yields (2), since $0 < x_i \leq c$ it follows that $0 < a \leq c$.

Levinson's operator inequality

Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. We denote by $\mathcal{B}_h(H)$ the real subspace of all self-adjoint operators on H.

A continuous real valued function f defined on an interval I is said to be operator convex if $f(\lambda X + (1-\lambda)Y) \leq \lambda f(X) + (1-\lambda)f(Y)$ for all self-adjoint operators X, Y with spectra contained in I and all $\lambda \in [0,1]$.

If the function f is operator convex, then so-called Jensen's operator inequality $f(\Phi(X)) \leq \Phi(f(X))$ holds for any unital positive linear mapping Φ on $\mathcal{B}(H)$ and any $X \in \mathcal{B}_h(H)$ with spectrum contained in I. Many other versions of Jensen's operator inequality can be found in

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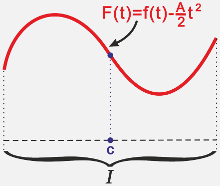
- T. Fujii, J. Mićić Hot, J. Pečarić and Y. Seo, *Recent Developments of Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 4, Element, Zagreb, 2012.

Definition 1.

Let $f \in \mathcal{C}(I)$ be a real valued functions on an arbitrary interval I in $\mathbb{R}$ and $c \in I^\circ$, where I° is the interior of I.

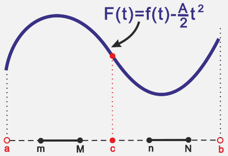
We say that $f \in \mathcal{K}_1^c(I)$ (resp. $f \in \mathcal{K}_2^c(I)$) if there exists a constant A such that the function $F(t) = f(t) - \frac{A}{2}t^2$ is concave (resp. convex) on $I \bigcap (-\infty, c]$ and convex (resp. concave) on $I \bigcap [c, \infty)$.



Moreover, we say that $f \in \dot{\mathcal{K}}_1^c(I)$ (resp. $f \in \dot{\mathcal{K}}_2^c(I)$) if F is operator concave (resp. operator convex) on $I \cap (-\infty, c]$ and operator convex (resp. operator concave) on $I \cap [c, \infty)$.

Theorem 1.

Let X and Y be self-adjoint operators $X, Y \in \mathcal{B}_h(H)$ with spectra contained in $[m, M]$ and $[n, N]$, respectively, such that $a < m \leq M \leq c \leq n \leq N < b$. Let Φ, Ψ be unital positive linear mappings $\Phi, \Psi : \mathcal{B}(H) \to \mathcal{B}(K)$.



If $f \in \dot{\mathcal{K}}_1^c((a,b))$ and $C_1 \leq C_2$, where

$$C_1 := \frac{A}{2}\left[\Phi(X^2) - \Phi(X)^2\right],\ C_2 := \frac{A}{2}\left[\Psi(Y^2) - \Psi(Y)^2\right], \tag{5}$$

then

$$\Phi(f(X)) - f(\Phi(X)) \leq C_1 \leq C_2 \leq \Psi(f(Y)) - f(\Psi(Y)). \tag{6}$$

But, if $f \in \dot{\mathcal{K}}_2^c((a,b))$ and $C_1 \geq C_2$ holds, then reverse inequalities are valid in (6).

Proof.

Let $f \in \overset{\bullet}{\mathcal{K}}_1^c((a,b))$. *So there is a constant A such that* $F(t) = f(t) - \frac{A}{2}t^2$ *is operator concave on* $(a,c]$. *Since* $[m,M] \subset (a,c]$, *then Jensen's operator inequality implies*

$$0 \leq F(\Phi(X)) - \Phi\big(F(X)\big) = f\big(\Phi(X)\big) - \frac{A}{2}\Phi(X)^2 - \Phi\big(f(X)\big) + \frac{A}{2}\Phi\big(X^2\big).$$

It follows

$$\Phi\big(f(X)\big) - f\big(\Phi(X)\big) \leq C_1. \tag{7}$$

Similarly, since F is operator convex on $[c,b)$ *and* $[n,N] \subset [c,b)$, *it follows*

$$C_2 \leq \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big). \tag{8}$$

Combining inequalities (7) *and* (8) *and taking into account that* $C_1 \leq C_2$ *we obtain desired inequality* (6).

Assume that $(\Phi_1, \ldots, \Phi_k)$ is a $k-$tuple of positive linear mappings $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$. If $\sum_{i=1}^{k} \Phi_i(1_H) = 1_K$, we say that $(\Phi_1, \ldots, \Phi_k)$ is *unital.*

Applying Theorem 1 we obtain more operators version of Levinson´s inequality.

Corollary 1.

Let $(X_1, \ldots, X_{k_1})$ be a k_1-tuple and $(Y_1, \ldots, Y_{k_2})$ be a k_2-tuple of self-adjoint operators $X_i, Y_j \in \mathcal{B}_h(H)$ with spectra contained in $[m, M]$ and $[n, N]$, respectively, such that $a < m \leq M \leq c \leq n \leq N < b$. Let $(\Phi_1, \ldots, \Phi_{k_1})$ be a unital k_1-tuple and $(\Psi_1, \ldots, \Psi_{k_2})$ be a unital k_2-tuple of positive linear mappings $\Phi_i, \Psi_j : \mathcal{B}(H) \to \mathcal{B}(K)$. If $f \in \dot{\mathcal{K}}_1^c((a, b))$ and

$$D_1 := \frac{A}{2}\left[\sum_{i=1}^{k_1} \Phi_i(X_i^2) - \left(\sum_{i=1}^{k_1} \Phi_i(X_i)\right)^2\right] \leq D_2 := \frac{A}{2}\left[\sum_{i=1}^{k_2} \Psi_i(Y_i^2) - \left(\sum_{i=1}^{k_2} \Psi_i(Y_i)\right)^2\right]$$

then

$$\sum_{i=1}^{k_1} \Phi_i(f(X_i)) - f\left(\sum_{i=1}^{k_1} \Phi_i(X_i)\right) \leq D_1 \leq D_2 \leq \sum_{i=1}^{k_2} \Psi_i(f(Y_i)) - f\left(\sum_{i=1}^{k_2} \Psi_i(Y_i)\right). \quad (9)$$

Corollary 1. (continued)

If $f \in \dot{\mathcal{K}}_2^c((a,b))$ and $D_1 \geq D_2$ holds, then reverse inequalities are valid in (9).

Corollary 1. (continued)

If $f \in \overset{\bullet}{\mathcal{K}}_2^c((a,b))$ and $D_1 \geq D_2$ holds, then reverse inequalities are valid in (9).

Proof. (▸)

We set
$$\tilde{\Phi}\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_{k_1} \end{pmatrix} = \sum_{i=1}^{k_1} \Phi_i(A_i), \qquad \tilde{\Psi}\begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_{k_2} \end{pmatrix} = \sum_{i=1}^{k_2} \Psi_i(B_i),$$
and
$$\tilde{X} = \begin{pmatrix} X_1 & & & 0 \\ & X_2 & & \\ & & \ddots & \\ 0 & & & X_{k_1} \end{pmatrix}, \qquad \tilde{Y} = \begin{pmatrix} Y_1 & & & 0 \\ & Y_2 & & \\ & & \ddots & \\ 0 & & & Y_{k_2} \end{pmatrix}.$$

Then $\tilde{\Phi}, \tilde{\Psi}$ are unital positive linear mappings, $\tilde{X} \in B_h(\underbrace{H \oplus \cdots \oplus H}_{k_1})$ and $\tilde{Y} \in B_h(\underbrace{H \oplus \cdots \oplus H}_{k_2})$ with spectra contained in $[m, M]$ and $[n, N]$, respectively. Applying Theorem 1 on these mappings and operators, we obtain desired inequality (9).

The following obvious corollary to Theorem 1 holds, with convex combinations of operators X_i, $i = 1, \dots, k_1$ and Y_i, $i = 1, \dots, k_2$.

Corollary 2.

Let $(X_1, \dots, X_{k_1})$ be a k_1–tuple and $(Y_1, \dots, Y_{k_2})$ be a k_2–tuple of self-adjoint operators $X_i, Y_j \in \mathcal{B}_h(H)$ with spectra contained in $[m, M]$ and $[n, N]$, respectively, such that $a < m \leq M \leq c \leq n \leq N < b$. Let $(p_1, \dots, p_{k_1})$ be a k_1–tuple and $(q_1, \dots, q_{k_2})$ be a k_2–tuple of positive scalars such that $\sum_{i=1}^{k_1} p_i = 1$ and $\sum_{i=1}^{k_2} q_i = 1$. If $f \in \dot{\mathcal{K}}_1^c((a, b))$ and

$$P := \frac{A}{2} \sum_{i=1}^{k_1} p_i \left(X_i - \bar{X}\right)^2 \leq Q := \frac{A}{2} \sum_{i=1}^{k_2} q_i \left(Y_i - \bar{Y}\right)^2$$

then

$$\sum_{i=1}^{k_1} p_i f(X_i) - f\left(\bar{X}\right) \leq P \leq Q \leq \sum_{i=1}^{k_2} q_i f(Y_i) - f\left(\bar{Y}\right), \tag{10}$$

where $\bar{X} = \sum_{i=1}^{k_1} p_i X_i$ and $\bar{Y} = \sum_{i=1}^{k_2} q_i Y_i$ denote the weighted arithmetic means of operators. But, if $f \in \dot{\mathcal{K}}_2^c([m_x, M_y])$ and $P_x \geq Q_y$ holds, then reverse inequalities are valid in (10).

Converse of Levinson's operator inequality

For convenience we introduce some abbreviations:

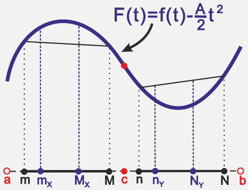
Let $f : [m, M] \to \mathbb{R}$, $m < M$, such that $F(t) = f(t) - \frac{A}{2}t^2$, $A \in \mathbb{R}$, be a convex or a concave function. We denote a linear function through $(m, F(m))$ and $(M, F(M))$ by $f^{line}_{A,[m,M]}$, i.e.

$$f^{line}_{A,[m,M]}(t) = \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M) - \frac{A}{2}\Big((M+m)t - mM\Big), \; t \in \mathbb{R}$$

and the slope and the intercept by $k_{A,f[m,M]}$ and $l_{A,f[m,M]}$, respectively, i.e.

$$\begin{aligned} k_{A,f[m,M]} &= \frac{f(M)-f(m)}{M-m} - \frac{A}{2}(M+m), \\ l_{A,f[m,M]} &= \frac{Mf(m)-mf(M)}{M-m} + \frac{A}{2}mM. \end{aligned}$$

Theorem 2. (▸)



Let X, Y m, M, n, N, Φ, Ψ, C_1, C_2 be as in Theorem 1. Let m_X, M_X, ($m_X \leq M_X$) and n_Y, N_Y, ($n_Y \leq N_Y$) be the bounds of operators $\Phi(X)$ and $\Psi(Y)$, respectively.

If $f \in \mathcal{K}_1^c((a,b))$ and $C_1 \geq C_2$, then

$$\Phi\left(f(X)\right) - f\left(\Phi(X)\right) + \beta_1 1_K \geq C_1 \geq C_2 \geq \Psi\left(f(Y)\right) - f\left(\Psi(Y)\right) + \beta_2 1_K, \tag{11}$$

where

$$\beta_1 = \max_{m_X \leq t \leq M_X} \left\{ f(t) - \frac{A}{2}t^2 - f^{line}_{A,[m,M]}(t) \right\} \geq 0, \tag{12}$$

$$\beta_2 = \min_{n_Y \leq t \leq N_Y} \left\{ f(t) - \frac{A}{2}t^2 - f^{line}_{A,[n,N]}(t) \right\} \leq 0. \tag{13}$$

Constants β_1, β_2 exist for any A, m,M,m_X,M_X and n,N,n_Y, N_Y.

Theorem 2. (continued)

The value $\beta_1 = f(t_0) - \frac{A}{2}t_0^2 - f^{line}_{A,[m,M]}(t_0)$*, where* t_0 *may be determined as follows:*

Theorem 2. (continued)

The value $\beta_1 = f(t_0) - \frac{A}{2} t_0^2 - f_{A,[m,M]}^{line}(t_0)$*, where* t_0 *may be determined as follows:*

- *if* $f'_-(t) - At \leq k_{A,f[m,M]}$ *for every* $t \in (m_x, M_x)$ *then* $t_0 = m_x$*,*

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The value $\beta_1 = f(t_0) - \frac{A}{2}t_0^2 - f^{line}_{A,[m,M]}(t_0)$*, where* t_0 *may be determined as follows:*

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- *if* $f'_-(t_1) - At_1 \geq k_{A,f[m,M]} \geq f'_+(t_1) - At_1$ *for some* $t_1 \in (m_x, M_x)$ *then* $t_0 = t_1$*,*

Theorem 2. (continued)

The value $\beta_1 = f(t_0) - \frac{A}{2}t_0^2 - f^{line}_{A,[m,M]}(t_0)$, *where* t_0 *may be determined as follows:*

- *if* $f'_-(t) - At \leq k_{A,f[m,M]}$ *for every* $t \in (m_x, M_x)$ *then* $t_0 = m_x$,
- *if* $f'_-(t_1) - At_1 \geq k_{A,f[m,M]} \geq f'_+(t_1) - At_1$ *for some* $t_1 \in (m_x, M_x)$ *then* $t_0 = t_1$,
- *if* $f'_+(t) - At \geq k_{A,f[m,M]}$ *for every* $t \in (m_x, M_x)$ *then* $t_0 = M_x$.

Theorem 2. (continued)

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- *if* $f'_-(t_1) - At_1 \geq k_{A,f[m,M]} \geq f'_+(t_1) - At_1$ *for some* $t_1 \in (m_x, M_x)$ *then* $t_0 = t_1$,
- *if* $f'_+(t) - At \geq k_{A,f[m,M]}$ *for every* $t \in (m_x, M_x)$ *then* $t_0 = M_x$.

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- *if* $f'_-(t) - At \leq k_{A,f[m,M]}$ *for every* $t \in (m_x, M_x)$ *then* $t_0 = m_x$,
- *if* $f'_-(t_1) - At_1 \geq k_{A,f[m,M]} \geq f'_+(t_1) - At_1$ *for some* $t_1 \in (m_x, M_x)$ *then* $t_0 = t_1$,
- *if* $f'_+(t) - At \geq k_{A,f[m,M]}$ *for every* $t \in (m_x, M_x)$ *then* $t_0 = M_x$.

The value of β_2 *can be determined as* β_1 *if we replace* m, M, m_x, M_x *by* n, N, n_y, N_y*, respectively, and with reverse inequality signs.*

Theorem 2. (continued)

The value $\beta_1 = f(t_0) - \frac{A}{2}t_0^2 - f^{line}_{A,[m,M]}(t_0)$*, where* t_0 *may be determined as follows:*

- *if* $f'_-(t) - At \le k_{A,f[m,M]}$ *for every* $t \in (m_x, M_x)$ *then* $t_0 = m_x$*,*
- *if* $f'_-(t_1) - At_1 \ge k_{A,f[m,M]} \ge f'_+(t_1) - At_1$ *for some* $t_1 \in (m_x, M_x)$ *then* $t_0 = t_1$*,*
- *if* $f'_+(t) - At \ge k_{A,f[m,M]}$ *for every* $t \in (m_x, M_x)$ *then* $t_0 = M_x$*.*

The value of β_2 *can be determined as* β_1 *if we replace* m, M, m_x, M_x *by* n, N, n_y, N_y*, respectively, and with reverse inequality signs.*

In the dual case, if $f \in \mathcal{K}_2^c((a,b))$ *and* $C_1 \le C_2$ *holds, then reverse inequalities are valid in* (11) *with* $\beta_1 \le 0$ *and* min *instead of* max *in* (12) *and* $\beta_2 \ge 0$ *and* max *instead of* min *in* (13)*. The value of constants* β_1 *and* β_2 *can be determined as above with reverse inequality signs.*

Proof.

We use the same technique as in the proof of Theorem 1 and converses of Jensen´s operator inequality are given in the paper

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*J.Mićić, Z.Pavić and J.Pečarić, Some better bounds in converses of the Jensen operator inequality, Oper. Matrices **6** (2012), 589-605.*

Proof.

We use the same technique as in the proof of Theorem 1 and converses of Jensen´s operator inequality are given in the paper

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Remark 1.

Let the assumptions of Theorem 2 be satisfied and $m < M$, $n < N$. *If* $C_1 \geq C_2$, *F is strictly concave differentiable on* $[m, c]$ *and strictly convex differentiable on* $[c, N]$, *then* (11) *holds, for*

$$\beta_1 = f(x_0) - \frac{A}{2}x_0^2 - f^{line}_{A,[m,M]}(x_0) \quad \leq \quad f(\bar{x}_0) - \frac{A}{2}\bar{x}_0^2 - f^{line}_{A,[m,M]}(\bar{x}_0),$$
$$\beta_2 = f(y_0) - \frac{A}{2}y_0^2 - f^{line}_{A,[n,N]}(y_0) \quad \geq \quad f(\bar{y}_0) - \frac{A}{2}\bar{y}_0^2 - f^{line}_{A,[n,N]}(\bar{y}_0),$$

where

Remark 1. (continued)

$$x_0 = \begin{cases} m_x & \textit{if } f'(m_A) - Am_A \le k_{A,f[m,M]}, \\ \text{the unique solution of the equation } f'(t) - At = k_{A,f[m,M]} \\ \quad \textit{if } f'(m_x) - Am_x \ge k_{A,f[m,M]} \ge f'(M_x) - AM_x, \\ M_x & \textit{if } f'(m_x) - AM_x \ge k_{A,f[m,M]}, \end{cases}$$

$\bar{x}_0$ is the unique solution in (m, M) of the equation $f'(t) - At = k_{A,f[m,M]}$;

y_0, $\bar{y}_0$ can be determined as x_0, $\bar{x}_0$, if we replace m,M,m_x,M_x by n,N,n_y, N_y, respectively, and with reverse inequality signs.

In the dual case, if $C_1 \le C_2$, f is strictly convex differentiable on $[m, c]$ and strictly concave differentiable on $[c, N]$, then reverse inequalities are valid in (11), with x_0, $\bar{x}_0$, y_0, $\bar{y}_0$ as above with reverse inequality signs.

Remark 2.

Let the assumptions of Theorem 2 be satisfied, $f \in \dot{\mathcal{K}}_1^c((a,b))$ *and* $C_1 \geq C_2$*. If F is operator concave, then* $0 \geq \Phi\big(F(X)\big) - F\big(\Phi(X)\big)$ *and LHS of* (11) *imply*

$$C_1 + \beta_1 1_K \geq \Phi\big(f(X)\big) - f\big(\Phi(X)\big) + \beta_1 1_K \geq C_1.$$

If F is operator convex, then $0 \leq \Phi\big(F(X)\big) - F\big(\Phi(X)\big)$ *and RHS of* (11) *imply*

$$C_2 \geq \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big) + \beta_2 1_K \geq C_2 + \beta_2 1_K.$$

It follows (an extension of (11)*)*

$$\begin{aligned} C_1 + \beta_1 1_K &\geq \Phi\big(f(X)\big) - f\big(\Phi(X)\big) + \beta_1 1_K \geq C_1 \geq C_2 \\ &\geq \Psi\big(f(Y)\big) - f\big(\Psi(Y)\big) + \beta_2 1_K \geq C_2 + \beta_2 1_K \end{aligned} \qquad (14)$$

But, if $f \in \dot{\mathcal{K}}_2^c((a,b))$ *and* $C_1 \leq C_2$*, then reverse inequalities are valid in* (14).

Applying Theorem 2
we can obtain converse of Levinson's operators inequality for k_1 and k_2-tuples of self-adjoint operators $X_i, Y_j \in \mathcal{B}_h(H)$ using the same technique as in the proof of Corollary 1.

We omit the details.

Refined Levinson's operator inequality

The absolute value of $B \in \mathcal{B}(H)$ is defined by $|B| = (B^* B)^{1/2}$.

For convenience we introduce abbreviations $\bar{\Delta}, \widetilde{\Delta}$ and δ as follows:

$$\bar{\Delta} \equiv \bar{\Delta}_{B,\Phi}(m,M) := \frac{1}{2}1_K - \frac{1}{M-m}\left|\Phi(B) - \frac{m+M}{2}1_K\right|, \qquad (15)$$

$$\widetilde{\Delta} \equiv \widetilde{\Delta}_{B,\Phi}(m,M) := \frac{1}{2}1_K - \frac{1}{M-m}\Phi\left(\left|B - \frac{m+M}{2}1_H\right|\right), \qquad (16)$$

where $B \in \mathcal{B}_h(H)$, Φ is an unital positive linear mapping, m, M, $m < M$, are some scalars such that spectra $\mathrm{Sp}(B) \subseteq [m,M]$;

$$\delta \equiv \delta_{f,A}(m,M) := 2f\left(\frac{m+M}{2}\right) - f(m) - f(M) + \frac{A}{4}(M-m)^2, \qquad (17)$$

where $f : [m,M] \to \mathbb{R}$ is a continuous function and $A \in \mathbb{R}$.

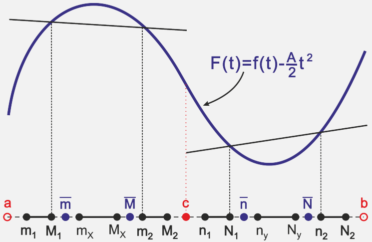
Obviously, $\bar{\Delta}, \widetilde{\Delta} \geq 0$. If $F(t) = f(t) - \frac{A}{2}t^2$ is *concave* (resp. *convex*) then $\delta \geq 0$ (resp. $\delta \leq 0$).

Next, we show refined Levinson's inequality given in Theorem 1 for two pairs of operators without operator concavity-convexity, and with spectra conditions.

Theorem 3.

Let $\Phi, \Psi : \mathcal{B}(H) \oplus \mathcal{B}(H) \to \mathcal{B}(K) \oplus \mathcal{B}(K)$ *be unital mappings such that* $\Phi\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \Phi_1(B_1) + \Phi_2(B_2)$ *and* $\Psi\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \Psi_1(B_1) + \Psi_2(B_2)$, *where* $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ *be positive linear mappings.*

Let $X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \in B_h(H \oplus H)$, *where* X_1, X_2, Y_1, Y_2 *be self-adjoint operators with spectra*



$\mathrm{Sp}(X_1) \subseteq [m_1, M_1]$, $\mathrm{Sp}(X_2) \subseteq [m_2, M_2]$,

$\mathrm{Sp}(Y_1) \subseteq [n_1, N_1]$, $\mathrm{Sp}(Y_2) \subseteq [n_2, N_2]$.

Let $M_1 < m_2$, $N_1 < n_2$ *and* $m_1 \le M_1 \le m_x \le M_x \le m_2 \le M_2 \le c \le n_1 \le N_1 \le n_y \le N_y \le n_2 \le N_2$, *where* m_x, M_x *be bounds of the operator* $\Phi(X)$ *and* n_y, N_y *be bounds of* $\Psi(Y)$.

Theorem 3. (continued)

If $f \in \mathcal{K}_1^c([m_1, N_2])$ and $C_1 \le C_2$,
then

$$\begin{aligned} & \Phi\left(f(X)\right) - f\left(\Phi(X)\right) \le \Phi\left(f(X)\right) - f\left(\Phi(X)\right) + \delta_1 \bar{X} \le C_1 \le C_2 \\ \le \; & \Psi\left(f(Y)\right) - f\left(\Psi(Y)\right) + \delta_2 \bar{Y} \le \Psi\left(f(Y)\right) - f\left(\Psi(Y)\right), \end{aligned} \tag{18}$$

where $\delta_1 = \delta_{f,A}(\bar{m}, \bar{M}) \ge 0$, $\bar{X} = \bar{\Delta}_{X,\Phi}(\bar{m}, \bar{M}) \ge 0$ for arbitrary numbers $\bar{m} \in [M_1, m_x], \bar{M} \in [M_x, m_2], \bar{m} < \bar{M}$

and $\delta_2 = \delta_{f,A}(\bar{n}, \bar{N}) \le 0$, $\bar{Y} = \bar{\Delta}_{Y,\Psi}(\bar{n}, \bar{N}) \ge 0$ for arbitrary numbers $\bar{n} \in [N_1, n_y], \bar{N} \in [N_y, n_2], \bar{n} < \bar{N}$.

But, if $f \in \mathcal{K}_2^c([m_1, N_2])$ and $C_1 \ge C_2$ holds, then reverse inequalities are valid in (18), with $\delta_1 \le 0$ and $\delta_2 \ge 0$.

Proof.

We will give the proof for $f \in \mathcal{K}_1^c([m_1, N_2])$*. Since* $F(t) = f(t) - \frac{A}{2}t^2$ *is concave on* $[m_1, c]$ *for some constant A, then*

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gives

$$F(\Phi(X)) \geq \Phi\left(F(X)\right) + \widetilde{\delta}_1 \bar{X} \geq \Phi\left(F(X)\right)$$

$$\Rightarrow \qquad C_1 \geq \Phi\left(f(X)\right) - f\left(\Phi(X)\right) + \delta_1 \bar{X} \geq \Phi\left(f(X)\right) - f\left(\Phi(X)\right), \qquad (19)$$

Similarly, since F is convex on $[c, N_2]$ *for some constant A, we have*

$$C_2 \leq \Psi\left(f(Y)\right) - f\left(\Psi(Y)\right) + \delta_2 \bar{Y} \leq \Psi\left(f(Y)\right) - f\left(\Psi(Y)\right). \qquad (20)$$

Combining inequalities (19) and (20) and taking into account $C_1 \leq C_2$, *we obtain desired inequality (18).*

Applying Theorem 3 we obtain more operators version of refined Levinson's inequality.

Corollary 3.

Let $(\Phi_1, \ldots, \Phi_{k_1})$ be a unital k_1-tuple and $(\Psi_1, \ldots, \Psi_{k_2})$ be a unital k_2-tuple of positive linear mappings $\Phi_i, \Psi_j : \mathcal{B}(H) \to \mathcal{B}(K)$.
Let $(X_1, \ldots, X_{k_1})$ be a k_1-tuple and $(Y_1, \ldots, Y_{k_2})$ be a k_2-tuple of self-adjoint operators X_i and $Y_j \in \mathcal{B}_h(H)$ with spectra contained in $[m_i, M_i]$ and $[n_j, N_j]$, respectively, such that

$$a < m_i \leq M_i \leq c \leq n_j \leq N_j < b,\ i = 1, \ldots, k_1,\ j = 1, \ldots, k_2,$$
$$(m_x, M_x) \cap [m_i, M_i] = \varnothing,\ i = 1, \ldots, k_1, \quad (m_y, M_y) \cap [n_j, N_j] = \varnothing,\ j = 1, \ldots, k_2,$$
$$m < M, \quad n < N$$

where m_x and M_x be bounds of $X = \sum_{i=1}^{k_1} \Phi_i(X_i)$, and n_y and n_y be bounds of $Y = \sum_{i=1}^{k_2} \Psi_i(Y_i)$; $m := \max\{M_i | M_i \leq m_x, i = 1, \ldots, k_1\}$, $M := \min\{m_i | m_i \geq M_x, i = 1, \ldots, k_1\}$, $n := \max\{N_i | N_i \leq n_y, i = 1, \ldots, k_2\}$, $N := \min\{n_i | n_i \geq N_y, i = 1, \ldots, k_2\}$.
If $f \in \mathcal{K}_1^c((a, b))$ and $D_1 \leq D_2$

Corollary 3. (continued)

then

$$\sum_{i=1}^{k_1} \Phi_i(f(X_i)) - f\Big(\sum_{i=1}^{k_1} \Phi_i(X_i)\Big) \leq \sum_{i=1}^{k_1} \Phi_i(f(X_i)) - f\Big(\sum_{i=1}^{k_1} \Phi_i(X_i)\Big) + \delta_1 \bar{X}$$
$$\leq D_1 \leq D_2 \tag{21}$$
$$\leq \sum_{i=1}^{k_2} \Psi_i(f(Y_i)) - f\Big(\sum_{i=1}^{k_2} \Psi_i(Y_i)\Big) \leq \sum_{i=1}^{k_2} \Psi_i(f(Y_i)) - f\Big(\sum_{i=1}^{k_2} \Psi_i(Y_i)\Big) + \delta_2 \bar{Y},$$

where $\delta_1 = \delta_{f,A}(\bar{m}, \bar{M}) \geq 0$, $\bar{X} = \bar{\Delta}_X(\bar{m}, \bar{M}) \geq 0$ *for arbitrary numbers* $\bar{m} \in [m, m_x], \bar{M} \in [M_x, M], \bar{m} < \bar{M}$ *and* $\delta_2 = \delta_{f,A}(\bar{n}, \bar{N}) \leq 0$, $\bar{Y} = \bar{\Delta}_Y(\bar{n}, \bar{N}) \geq 0$ *for arbitrary numbers* $\bar{n} \in [n, n_y], \bar{N} \in [N_y, N], \bar{n} < \bar{N}$.

In above we denote

$$\bar{\Delta} \equiv \bar{\Delta}_B(m, M) := \frac{1}{2} 1_K - \frac{1}{M-m} \Big| B - \frac{m+M}{2} 1_K \Big|.$$

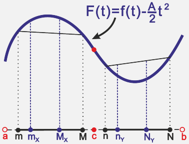
But, $f \in \mathcal{K}_2^c((a,b))$ *and* $D_1 \geq D_2$ *holds, then reverse inequalities are valid in* (21), *with* $\delta_1 \leq 0$ *and* $\delta_2 \geq 0$.

Refined converse of L. operator inequality

Now, we give refined converse of Levinson's operator inequality given in Theorem 2 for two pairs of operators.

Theorem 4.

Let Φ, Ψ be mappings and , X, Y be operators as in Theorem 3, with spectra $\mathrm{Sp}(X_1), \mathrm{Sp}(X_2) \subseteq [m, M]$, $\mathrm{Sp}(Y_1), \mathrm{Sp}(Y_2) \subseteq [n, N]$. Let m_X, M_X be bounds of the operator $\Phi(X)$ and n_Y, N_Y be bounds of $\Psi(Y)$.



If $f \in \mathcal{K}_1^c([m, N])$ and $C_1 \geq C_2$, then

$$
\begin{aligned}
& \Phi(f(X)) - f(\Phi(X)) + \beta_1 1_K \geq \Phi(f(X)) - f(\Phi(X)) + \beta_1 1_K - \delta_1 \widetilde{X} \\
\geq \; & C_1 \geq C_1 - \delta_1 \widetilde{X} \geq C_2 - \delta_2 \widetilde{Y} \geq C_2 \\
\geq \; & \Psi(f(Y)) - f(\Psi(Y)) + \beta_2 1_K - \delta_2 \widetilde{Y} \geq \Psi(f(Y)) - f(\Psi(Y)) + \beta_2 1_K,
\end{aligned}
\tag{22}
$$

where β_1 and β_2 are defined in Theorem 2, $\delta_1 = \delta_{f,A}(m, M) \geq 0$, $\widetilde{X} = \widetilde{\Delta}_{X,\Phi}(m, M) \geq 0$, $\delta_2 = \delta_{f,A}(n, N) \leq 0$ and $\widetilde{Y} = \widetilde{\Delta}_{Y,\Psi}(n, N) \geq 0$.

But, If $f \in \mathcal{K}_2^c([m, N])$ and $C_1 \leq C_2$ holds, then reverse inequalities are valid in (22), with $\delta_1 \leq 0$ and $\delta_2 \geq 0$.

Proof.

We use the same technique as in the proof of Theorem 3 and converses of Jensen´s operator inequality are given in the paper

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Remark 3.

Let the assumptions of Theorem 4 be satisfied, $f \in \overset{\bullet}{\mathcal{K}}{}_1^c((a,b))$ *and* $C_1 \geq C_2$*. Then the following extension of (22) holds*

$$\begin{aligned} & C_1+\beta_1 1_K \geq C_1+\beta_1 1_K-\widetilde{\delta}_1\widetilde{X} \\ \geq\; & \Phi\big(f(X)\big)-f\big(\Phi(X)\big)+\beta_1 1_K-\widetilde{\delta}_1\widetilde{X} \\ \geq\; & C_1 \geq C_1-\delta_1\widetilde{X} \geq C_2-\widetilde{\delta}_2\widetilde{Y} \geq C_2 \\ \geq\; & \Psi\big(f(Y)\big)-f\big(\Psi(Y)\big)+\beta_2 1_K-\delta_2\widetilde{Y} \\ \geq\; & C_2+\beta_2 1_K-\widetilde{\delta}_2\widetilde{Y} \geq C_2+\beta_2 1_K \end{aligned} \tag{23}$$

If $f \in \overset{\bullet}{\mathcal{K}}{}_2^c((a,b))$ *and* $C_1 \leq C_2$*, then reverse inequalities are valid in (23).*

Applying Theorem 4 we can obtain converse of Levinson´s operator inequality for k_1 and k_2-tuples of self-adjoint operators $X_i, Y_j \in \mathcal{B}_h(H)$ (with weakened assumptions than in Corollary 3).

We omit the details.

Thank you

for your attention