Determinants in K-theory and multivariable operator theory

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Definition

Let $a, b \in \mathcal{B}/\mathcal{L}^1$ be invertible commuting elements. The **determinant invariant** $d(a, b) = \det \partial\{a, b\} \in \mathbf{C}^\times$.

Examples

Theorem (Brown, '75)

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$$d(T_{e^f} + \mathcal{L}^1, T_{e^g} + \mathcal{L}^1) = \exp \frac{1}{2\pi i} \int f\, dg.$$

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Theorem (Kaad, '11)

$d(a, b)$ is the Connes-Karoubi multiplicative character of $\{a, b\}$.

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Torsion

Let $(V_\bullet, d_\bullet)$ be an exact sequence of finite dimensional vector spaces:

$$0 \to V_n \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} V_{n-2} \to \cdots \to V_0 \to 0$$

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Denote by $\det V_k$ the top exterior power $\Lambda^{\dim V_k} V_k$ and define

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(For each k, pick a nonzero element $t_k \in \Lambda^{\mathrm{rank}\, d_k} V_k$ such that $d_k t_k \neq 0$. By exactness, $d_k t_k \wedge t_{k-1} \in \det V_{k-1}$ is nonzero. Then let

$$\tau(V_\bullet, d_\bullet) = (t_n)^* \otimes (d_n t_n \wedge t_{n-1}) \otimes (d_{n-1} t_{n-1} \wedge t_{n-2})^* \otimes \dots$$

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and hence a long exact sequence in homology:

$$\mathcal{E}_B : \cdots \to H_{i+1}(A, B) \to H_i(A) \to H_i(A) \to H_i(A, B) \to \cdots$$

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Switching the roles of A and B, we obtain

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Definition

The **joint torsion** $\tau(A, B)$ of two Fredholm operators is (up to a sign) the nonzero scalar

$$\tau(A, B) = \tau(\mathcal{E}_A) \otimes \tau(\mathcal{E}_B)^*$$

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Conjecture

$\tau(A, B, C, D) = d(a, b)$.

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Group actions

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Theorem (JM '14)

$$\tau\left((U, V) \bullet (A, B, C, D)\right) = d(a, \pi(U)) \cdot \tau(A, B, C, D)$$

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By combining this result with the previous slide, the following theorem, that joint torsion equals the determinant invariant, follows quickly.

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Theorem (JM '14)

Joint torsion equals the determinant invariant, that is,

$$\tau(A, B, C, D) = d(a, b)$$

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- $\tau(a, b)$ satisfies the usual Steinberg relations, i.e. skew-symmetry, bimultiplicativity, etc.

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  endowed with the metric

$$d((A_1, B_1), (A_2, B_2)) = \|A_1 - A_2\| + \|B_1 - B_2\| \\ + \|[A_1, B_1] - [A_2, B_2]\|_1$$

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- $d(a, b)$ can be computed in terms of finite dimensional data.

Local properties

Theorem (Eschmeier-Putinar, '96)

Suppose A is a Fredholm n-tuple, $g \in \mathcal{O}(Sp(A))^n$, and $g(A)$ is Fredholm. Then

$$ind\, g(A) = \sum_{\{\lambda \in Sp(A)\,|\,g(\lambda)=0\}} deg_\lambda(g) \cdot ind(A - \lambda)$$

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- If $f, g \in H^\infty$, denote by $c_\lambda(f, g)$ the tame symbol of f and g.

$$\tau(T_f, T_g) = \prod_{|\lambda|<1} c_\lambda(f, g)$$

Further work

- Joint torsion of n-tuples for $n > 2$
  - Kaad, '12: commuting case
  - JM '13: perturbation vectors
  - Kaad-Nest '14: analyticity, localized torsion, tame symbols
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- Geometric examples: Toeplitz operators, pseudodifferential operators...

Thank you

very much