Extensions of integral type operators on Hardy spaces

joint work with B. Staniów (AMU Poznań)

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Hardy spaces

Let $f \in H(\mathbb{D}), \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, be a holomorphic function on a unit disc.

For $1 \leqslant p \leqslant \infty, 0 \leqslant r < 1$ we define

$$M_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, \mathrm{d}t$$
$$M_\infty(r, f) = \sup_{t \in [0, 2\pi)} |f(re^{it})|$$

A function $f \in H(\mathbb{D})$ belongs to the Hardy space $H^p = H^p(\mathbb{D})$, if

$$\sup_{0 \leqslant r < 1} M_p(r, f) < \infty$$

If $1 \leqslant p \leqslant q \leqslant \infty$, then $H^q \subset H^p$.

Integral operator

For a holomorphic function g on the unit disc $\mathbb{D}$, an integral operator T_g is defined by

$$T_g f(z) = \frac{1}{z} \int_0^z f(\xi) g'(\xi)\, d\xi, \quad f \text{ holomorphic on the disc}, \ z \in \mathbb{D}.$$

T_g is bounded on H^p ($p \in [1, \infty)$) if and only if $g \in BMOA$.

However, $T_g : H^p \to H^p$ is not “onto”, i.e. there exists a function $f \notin H^p$ such that $T_g f \in H^p$.

Basic observation

Proposition

Let $g \in H(\mathbb{D})$ satisfies $g'(z) \neq 0$, $z \in \mathbb{D}$, and let's identify two such functions if they differ only by a constant. Then the integral operator

$$T_g f(z) = \frac{1}{z} \int_0^z f(\xi) g'(\xi)\, d\xi, \quad z \in \mathbb{D}$$

is an isomorphism of $H(\mathbb{D})$ onto $H(\mathbb{D})$.

In consequence there exists a function $f \in H(\mathbb{D}) \setminus H^p$ such that $T_g f \in H^p$.

Basic observation

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Let $g \in H(\mathbb{D})$ *satisfies* $g'(z) \neq 0$, $z \in \mathbb{D}$, *and let's identify two such functions if they differ only by a constant. Then the integral operator*

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is an isomorphism of $H(\mathbb{D})$ *onto* $H(\mathbb{D})$.

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Problem

Does there exists a Banach space of analytic functions on $\mathbb{D}$ *(larger than* H^p*) which* $\mathscr{C}$ *maps continuously into* H^p*? Does there exists the largest such space (i.e. optimal domain space)?*

Cesàro operator

If $g'(z) = (1-z)^{-1}$, $z \in \mathbb{D}$, then the integral operator T_g is the Cesàro operator $\mathscr{C}$.

This problem of describing the optimal domain space of $\mathscr{C}$ was answered in 2008 by Curbera and Ricker.

Integral op. – extension within Hardy spaces

Theorem (Aleman and Cima, 2001)

Let $p, q > 0$. Then we have the following:

- *For $q > p$, T_g maps H^q into H^p if and only if $g \in H^s$, where $1/s = 1/p - 1/q$.*
- *T_g maps H^p into itself if and only if $g \in BMOA$.*
- *For $q < p$ and $1/q - 1/p \leqslant 1$, T_g maps H^q into H^p if and only if $g \in \Lambda_{1/q-1/p}$.*
- *If $1/q - 1/p > 1$, and T_g maps H^q into H^p, then g is constant.*

The symbol Λ_α, $\alpha \in (0, 1]$, stands for the "big oh" analytic Lipschitz class, meaning $g \in \Lambda_\alpha$ if and only if $|g'(z)| = \mathcal{O}((1 - |z|)^{\alpha-1})$, $z \in \mathbb{D}$.

P-admissible functions

A function $g \in H(\mathbb{D})$ is called p-admissible if $g \in BMOA$, $g'(z) \neq 0$ for $z \in \mathbb{D}$ and additionally (a technical assumption) g' has a radial limit a.e. on $\mathbb{T}$.

Weighted Hardy spaces

Let us recall that for a weight ω, that is for any function such that $\omega > 0$ and $\log \omega$ is integrable, and an outer function ψ corresponding to ω, the weighted Hardy space $H^p(\omega)$ consists of $f \in H(\mathbb{D})$ such that $\psi^{1/p} f \in H^p$ and is equipped with a norm $\|f\|_{H^p(\omega)} = \|\psi^{1/p} f\|_p$.

Extension of T_g within weighted Hardy spaces

Theorem

Let $p \in [1, \infty)$ and ω be a weight with ψ an outer function corresponding to ω. The following conditions are equivalent:

- $T_g \colon H^p(\omega) \to H^p$ *continuously.*
- *Operator* $T \colon H^p \to H^p$ *given for all* $h \in H^p$ *and* $z \in \mathbb{D}$ *by the formulae*
$$Th(z) = \int_0^z h(\xi)\psi^{-1/p}(\xi)g'(\xi)d\xi$$
is continuous.
- *Function* $\rho_\psi \colon \mathbb{D} \to \mathbb{D}$ *given by the formulae*
$$\rho_\psi(z) = \int_0^z \psi^{-1/p}(\xi)g'(\xi)d\xi, \quad z \in \mathbb{D},$$
belongs to BMOA.

Extension of T_g within weighted Hardy spaces

Corollary

Let ψ be an outer function corresponding to ω. If $g \in$ LMOA and $\psi^{-1/p} \in$ BMOA then T_g maps $H^p(\omega)$ into H^p continuously.

LMOA consists of such $f \in VMOA$ that

$$\lim_{|I|\to 0} \frac{\log \frac{1}{|I|}}{|I|} \int_I |f - f_I| = 0.$$

Optimal domain space of an integral operator

Proposition

Let $p \in [1, \infty)$ and let g be a p-admissible function. Then the following holds

$$[T_g, H^p] = \left\{ f \in H(\mathbb{D}) : f(z) = \frac{h'(z)}{g'(z)},\ h \in H^p \right\}. \tag{1}$$

$[T_g, H^p]$ is a Banach space when equipped with a norm $\| \cdot \|_{[T_g, H^p]} := \|T_g \cdot \|_{H^p}$.

Example

Let g be a anti-derivative of a function

$$g'(z) = \frac{1}{1-z} + \sum_{i=2}^{n} \frac{\alpha_i}{1-\beta_i z},$$

$$z \in \mathbb{D},\ \alpha_i \neq 0,\ \beta_i \neq \beta_j,\ i \neq j,\ i,j = 2,\ldots,n.$$

Then

- T_g is a bounded operator on H^p,
- The domain of T_g can not be extended to H^q for any $q < p$.

Optimal domain – properties

Proposition

For any $p \in [1, \infty)$, $H^p \subsetneq [T_g, H^p]$.

Lemma

Point evaluations are continuous on $[T_g, H^p]$, $1 \leqslant p < \infty$.

Theorem

Let $p \in [1, \infty)$ *and let* $g \in H(\mathbb{D})$ *with* $\frac{1}{g'} \in H^\infty$. *Then for* $\phi \in H(\mathbb{D})$ *the multiplication operator* $M_\phi(f) := \phi \cdot f$ *is well defined and continuous on* $[T_g, H^p]$ *if and only if* $\phi \in H^\infty$.

Libera operator

For a point $z_0 \in \mathbb{C}$, $|z_0| \leqslant 1$, and a function $f \in H(\mathbb{D})$ we formally define the Libera operator by

$$\mathscr{L}_{z_0} f(z) = \frac{1}{z - z_0} \int_{z_0}^{z} f(\xi)\, d\xi, \quad z \in \mathbb{D}.$$

Optimal domain of the Libera operator $\mathscr{L}_0$

Corollary

Let $p \in [1, \infty)$ and ω be a weight with ψ being an outer function corresponding to ω. The following conditions are equivalent:

- $\mathscr{L}_0 \colon H^p(\omega) \to H^p$
- *Operator $T \colon H^p \to H^p$ given for $f \in H^p$ with the formula*

$$Tf(z) = \int_0^z f(\xi)\psi^{-1/p}(\xi)\, d\xi, \quad z \in \mathbb{D}, \tag{2}$$

  is continuous.
- *Function $\rho_\psi \colon \mathbb{D} \to \mathbb{D}$ given by*

$$\rho_\psi(z) = \int_0^z \psi^{-1/p}(\xi)\, d\xi \tag{3}$$

  belongs to $BMOA$.

Corollary

Corollary

Let $p \in [1, \infty)$. *Then*

- $[\mathscr{L}_0, H^p] = \{f \in H(\mathbb{D}) : f(z) = h'(z),\ h \in H^p\}$.
- $H^p \subsetneq [\mathscr{L}_0, H^p]$.

Corollary

Proposition

For $1 \leqslant p < \infty$:

- *$[\mathscr{L}_0, H^p]$ is separable.*
- *$[\mathscr{L}_0, H^p]$ is uniformly convex for $p \neq 1$.*
- *For $p = 2$, $[\mathscr{L}_0, H^2]$ is a Hilbert space and*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in [\mathscr{L}_0, H^p] \quad \textit{if and only if} \quad \left(\frac{a_n}{n+1}\right) \in \ell^2.$$

- *For $1 \leqslant p_1 < p_2 < \infty$ we have $[\mathscr{L}_0, H^{p_1}] \subset [\mathscr{L}_0, H^{p_2}]$.*
- *Polynomials are dense in $[\mathscr{L}_0, H^p]$.*

Libera operator $\mathscr{L}_1$

- $\mathscr{L}_1$ is well defined on $H(\overline{\mathbb{D}})$,
- $\mathscr{L}_1$ can not be extended to an operator from $H(\mathbb{D})$,
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Theorem (Nowak & Pavlović 2010, P.M. & Staniów 2014)

If $1 < p < \infty$ then $\mathscr{L}: H^p \to H^p$ is an adjoint of the Cesàro operator $\mathscr{C}: H^p \to H^p$ with a Cauchy pairing

$$\langle f, g \rangle = \lim_{r \to 1^-} \int_0^{2\pi} f(re^{it}) g(re^{-it})\, dt.$$

Libera operator $\mathscr{L}_1$ – optimal domain

Proposition

For each $1 < p < \infty$ *we have, as vector spaces, that*

$$[\mathscr{L}, H^p] = \{f \in H(\mathbb{D}) : f(z) = ((z-1)g(z))', g \in H^p\}.$$

Proposition

For each $1 < p < \infty$ *we have, that* $H^p \subsetneq [\mathscr{L}, H^p] \subsetneq [\mathscr{C}, H^p]$.

Thank you for your attention!