Hyperreflexivity of subspaces of Toeplitz operators on regions in the complex plane

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Joint work with Marek Ptak

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International Workshop on Operator Theory
Belfast, September 3-5, 2014

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Definition

The *Hardy space $H^2(\Omega)$* on Ω is the set of all analytic functions $F\colon \Omega \to \mathbb{C}$ for which there exists a function u, harmonic on Ω, such that

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By $H^\infty(\Omega)$ we denote the space of all bounded analytic functions on Ω.

The spaces $H^2(\Omega)$ and $H^\infty(\Omega)$ are Banach spaces with the norm

$$\|F\|_{H^2(\Omega)} := (u_F(a))^{1/2}, \quad \|F\|_{H^\infty(\Omega)} := \sup_{z\in\Omega} |F(z)|,$$

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Theorem (Rudin 1955)

$H^2(\Omega) \ni F \mapsto F^* \in H^2(\partial\Omega) \subset L^2(\partial\Omega)$

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Theorem (Paley, Wiener 1934)

$H^2(\mathbb{C}_+) \ni F \mapsto F^* \in H^2(\mathbb{R}) \subset L^2(\mathbb{R})$

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(b) *the subspace* $\mathcal{S}$ *is* k*-hyperreflexive with constant* c *iff* $U\mathcal{S}U^{-1}$ *is* k*-hyperreflexive with constant* c.

Theorem (Sarason 1966)

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Suppose that $\mathcal{B} \subset \mathcal{T}(\mathbb{D})$ is a weak closed subspace. Then the following statements are equivalent.*

(1) *$\mathcal{B}$ is not transitive.*

(2) *There is a function $f : \mathbb{T} \to \mathbb{C}$ such that $f \in L^1(\mathbb{T})$, $\log|f| \in L^1(\mathbb{T})$ and $\int_{\mathbb{T}} \varphi f \, dm = 0$ for all $T_\varphi \in \mathcal{B}$.*

(3) *$\mathcal{B}$ is reflexive.*

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Theorem (Davidson 1987 and Kliś, Ptak 2006)

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Theorem (Kliś, Ptak 2006)

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- If $\widetilde{U}_2(A) := U_2 A U_2^{-1}$, $A \in \mathcal{B}(H^2(\mathbb{D}))$, then
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Sketch of the proof

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  Moreover, $\widetilde{U}_2(\mathcal{T}(\mathbb{D})) = \mathcal{T}(\mathbb{C}_+)$ and $\widetilde{U}_2(\mathcal{A}(\mathbb{D})) = \mathcal{A}(\mathbb{C}_+)$.
- Finally, combining facts on hyperreflexivity of Toeplitz operators on $H^2(\mathbb{D})$ with Lemma on (hyper)reflexivity and spatially isomorphic subspaces we get the proof.

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