On joint numerical radius

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$$W(T_1, \dots, T_n) = \{(\langle T_1 x, x\rangle, \dots, \langle T_n x, x\rangle) : x \in H, \|x\| = 1\}.$$

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Problem

Let $T_1, \dots, T_n \in B(H)$. Does there exist $x \in H$ $\|x\| = 1$ such that $|\langle T_j x, x\rangle|$ is "large" for all $j = 1, \dots, n$?

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(c) We may assume $\|T_j\| = 1$.

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Let $T_1, \dots, T_n \in B(H)$, $T_j \geq 0 \quad (j = 1, \dots, n)$. *Let* $\dim H < \infty$. *Then there exists* $x \in H$, $\|x\| = 1$ *such that*

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$$|\langle T_j x, x\rangle| \geq \frac{1}{n}\|T_j\| \qquad (j = 1, \dots, n)$$

Theorem

Let $T_1, T_2 \in B(H)$, $T_j^* = T_j \quad (j = 1, 2)$, $\dim H < \infty$. *Then there exists* $x \in H$, $\|x\| = 1$ *such that*

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(the estimates are the best possible).

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Let Q be a convex subset of $[-1,1]^n$. *Let* $x^{(1)},\dots,x^{(n)} \in Q$ *satisfy* $x_k^{(k)} = 1 \quad (k=1,\dots,n)$.

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known: there exits $y \in Q$ with $|y_k| \ge \frac{1}{2n\sqrt{n}}$ for all k

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(3) Let $\mathcal{A}$ be a unital Banach algebra, $a_1, \dots, a_n \in \mathcal{A}$. Define the algebraic numerical range

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We have

$$\langle T_j x, y\rangle = \frac{1}{4}\Big(\langle T_j(x+y), x+y\rangle - \langle T_j(x-y), x-y\rangle$$

$$+i\langle T_j(x+iy), x+iy\rangle - i\langle T_j(x-iy), x-iy\rangle\Big).$$

Theorem

Let $T_1, \dots, T_n \in B(H)$. Then there exists $x \in H$, $\|x\| = 1$ such that

$$|\langle T_j x, x\rangle| \geq \frac{const}{n^2} \|T_j\| \qquad (j = 1, \dots, n).$$