# An asymmetric Putnam-Fuglede theorem for paranormal and $*$-paranormal operators

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*Let $A, B \in \mathcal{B}(\mathcal{H})$ be normal operators. If $AX = XB$, then $A^*X = XB^*$ for any $X \in \mathcal{B}(\mathcal{H})$.*

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How to relax the assumptions?

# Generalization of Putnam-Fuglede Theorem

## Example

*Let $S$ be a unilateral shift i.e.*

$$S : l^2 \ni (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots) \in l^2.$$

*The operator $S$ is subnormal (it can be extended to normal operator). The Putnam-Fuglede Theorem for $A = B = S$ does not hold. Indeed, for $X = S$ we have $SS = SS$, but $S^*S \neq SS^*$.*

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## Definition

*A N-PF class is a maximal (with respect to inclusion) class of operators $A$ such that $AX = XN$ implies $A^*X = XN^*$ for any $X \in \mathcal{B}(\mathcal{H})$ and normal $N \in \mathcal{B}(\mathcal{H})$.*

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An operator $T \in \mathcal{B}(\mathcal{H})$ is dominant if for any $\lambda \in \mathbb{C}$ there is a constant $M_\lambda > 0$ such that

$$\|(T^* - \overline{\lambda})f\| \leq M_\lambda \|(T - \lambda)f\|, \qquad \text{for any } f \in \mathcal{H}.$$

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In particular, hyponormal operators are dominant.

# The maximal class of operators which satisfies an asymmetric Putnam-Fuglede Theorem.

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*The maximal (with respect to inclusion) class of operators which satisfies an asymmetric Putnam-Fuglede Theorem and contains normal operators will be called a PF class.*

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$(iii) \Rightarrow (i)$ First, let us observe that if an operator does not satisfy $(iii)$ then it does not satisfy Putnam-Fuglede Theorem. Thus we have to prove the following theorem. □

### Theorem

*Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be operators which satisfies (iii).*
*If* $AX = XB^*$*, then* $A^*X = XB$ *for any* $X \in \mathcal{B}(\mathcal{H})$*.*

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To prove it, let us remain the Berberian's construction.
Let us fix a Banach limit $\phi$.
The space $\mathcal{B} := \{\{x_n\}_{n=1}^{\infty} \subset \mathcal{H} : \sup_n \|x_n\| < \infty\}$ is a vector space with semi-inner product

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The mapping

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is an isometric $*$-representation, i.e.

$$(S+T)^\circ = S^\circ + T^\circ, \quad (\lambda T)^\circ = \lambda T^\circ$$

$$(ST)^\circ = S^\circ T^\circ, \quad (T^*)^\circ = (T^\circ)^*, \quad \|T\| = \|T^\circ\|.$$

Moreover, $T \geq 0$ if and only if $T^\circ \geq 0$.

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**Theorem (Berberian)**

*Let $T \in \mathcal{B}(\mathcal{H})$. Then $\sigma_{ap}(T) = \sigma_{ap}(T^\circ) = \sigma_p(T^\circ)$ and $\sigma(T^\circ) \subset \sigma(T)$.*

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- thus by $(iii)$ the operator $A_{11}^\circ$ is diagonal and $A_{11}$ is normal,
- analogy $B_{11}$ is normal,
- by $(iii)$ the operators $A_{12}$ and $B_{12}$ are trivial,
- by classical Putnam-Fuglede Theorem we finised the proof.

# An asymmetric Putnam-Fuglede theorem for $*$-paranormal operators

We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is $*$-paranormal iff

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**Corollary**

*Let $A, B \in \mathcal{B}(\mathcal{H})$ be $*$-paranormal operators. If $AX = XB$, then $A^*X = XB^*$ for any $X \in \mathcal{B}(\mathcal{H})$.*

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Thank you for your attention!