Extensions of $B(E)$ for Banach spaces E

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Joint work with N.J.Laustsen

Talk Outline

- Extensions of Banach algebras
- Read's Banach Space
- Results

Extensions of Banach algebras

Definition

Let B be a Banach algebra. An *extension* of B is a short exact sequence of Banach algebras and continuous algebra homomorphisms:

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The extension *splits algebraically* if there is an algebra homomorphism $\rho : B \to A$ such that $\pi \circ \rho = \mathrm{id}_B$, and *splits strongly* if this map can be chosen to be continuous.

Extensions of Banach algebras

Questions about splittings (Bade, Dales, Lykova, '99):

1. For which Banach algebras is it true that every extension must split, either algebraically or strongly?
2. For which Banach algebras is it true that every extension which splits algebraically also splits strongly?

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1. For which Banach algebras is it true that every extension must split, either algebraically or strongly?
2. For which Banach algebras is it true that every extension which splits algebraically also splits strongly?
3. Is there an extension of $B(E)$ which splits algebraically but not strongly, for some Banach space E?

Theorem (Johnson, '67)

Let E be a Banach space. If $E \cong E \oplus E$ then every homomorphism from $B(E)$ into a Banach algebra is continuous.

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Corollary (Bade, Dales, Lykova, '99)

Let E be a Banach space. If $E \cong E \oplus E$ then every extension of $B(E)$ which splits algebraically also splits strongly.

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Banach spaces with discontinuous homomorphisms from $B(E)$:

- Read's space $E_{\mathcal{R}}$
- Dales-Loy-Willis' space E_{DLW} (with CH)

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How to approach this question? Need a Banach space such that:

- There is a discontinuous homomorphism from $B(E)$ into a Banach algebra,
- This Banach algebra is an extension of $B(E)$,
- There is no continuous homomorphism that splits the extension.

Read's Space

Theorem (Read,'89)

There is a Banach space $E_{\mathcal{R}}$ such that there is a discontinuous homomorphism from $B(E_{\mathcal{R}})$ into a Banach algebra.

$E_{\mathcal{R}}$ is a direct sum of quasi-reflexive, 'James-like' spaces.

Results

Theorem (Laustsen-S)

There exists a continuous surjective algebra homomorphism φ *such that the extension*

$$0 \longrightarrow W(E_{\mathcal{R}}) \xrightarrow{\iota} B(E_{\mathcal{R}}) \xrightarrow{\varphi} \tilde{\ell}_2 \longrightarrow 0$$

splits strongly.

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Corollary

The Banach algebra $B(E_{\mathcal{R}})$ *has the form*

$$B(E_{\mathcal{R}}) = W(E_{\mathcal{R}}) \oplus \rho(\ell_2) \oplus \mathbb{C}1$$

where $\rho : \tilde{\ell}_2 \to B(E_{\mathcal{R}})$ *is the strong splitting, so that* $W(E_{\mathcal{R}})$ *is complemented as a Banach space in* $B(E_{\mathcal{R}})$*.*

Results

Theorem (Laustsen-S)

Let B be a unital Banach algebra containing a proper closed (two-sided) ideal W such that $B = D \oplus \mathbb{C}1 \oplus W$ as a Banach space, where

(i) *D is a closed subspace of B,*

(ii) *$D^2 \subseteq W$,*

(iii) *$D \ncong \ell_1(J)$ for any index set J.*

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Proof.

We need to find W and D such that $B(E_{\mathcal{R}}) = D \oplus \mathbb{C}1 \oplus W$ and such that:

(i) D is a closed subspace of $B(E_{\mathcal{R}})$,

(ii) $D^2 \subseteq W$,

(iii) $D \not\cong \ell_1(J)$ for any index set J.

Take $D = \rho(\ell_2)$ and $W = W(E_{\mathcal{R}})$. (i) is the previous corollary. Conditions (ii) and (iii) follow since $\rho(\ell_2) \cong \ell_2$ as Banach algebras, where ℓ_2 has the trivial product. □

Results

What is the extension? We get:

$$0 \longrightarrow \mathrm{Ker}\ \psi \xrightarrow{\iota} \ell_1(\Gamma) \oplus \mathbb{C}1 \oplus W(E_{\mathcal{R}}) \xrightarrow{\psi} B(E_{\mathcal{R}}) \longrightarrow 0$$

where

- $\psi(f, \lambda 1, W) = q(f) + \lambda 1 + W$.
- $q : \ell_1(\Gamma) \to \rho(\ell_2)$ is a bounded linear surjection.
- Γ is a dense subset of the unit ball of $\rho(\ell_2)$.

Further Results

Definition

Let

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be an extension of a Banach algebra B. The extension is *radical* if $I \subset \mathrm{rad}\ B$, *singular* if $I^2 = 0$, and *admissible* if there is a continuous linear map $Q : B \to A$ such that $\pi \circ Q = \mathrm{id}_B$,.

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Proposition (Laustsen-S)

Let Σ be the extension of $B(E_R)$ given by the previous theorem. Then Σ is singular and radical, but is not admissible.

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