Subnormality via directed trees

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Operators

- By an **operator** in a complex Hilbert space $\mathcal{H}$ we mean a linear mapping $A\colon \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{H}$ defined on a vector subspace $\mathcal{D}(A)$ of $\mathcal{H}$, called the **domain** of A;
- A is said to be **normal** if A is densely defined, closed and

$$A^*A = AA^*;$$

equivalently: A is densely defined, $\mathcal{D}(A) = \mathcal{D}(A^*)$ and

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Particular classes of operators

- An operator S in $\mathcal{H}$ is **subnormal** if S is densely defined and there exists a complex Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Sh = Nh$ for every $h \in \mathcal{D}(S)$.
- An operator A in $\mathcal{H}$ is **hyponormal** if A is densely defined, $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^* f\| \leqslant \|Af\|$ for every $f \in \mathcal{D}(A)$.
- An operator A in $\mathcal{H}$ is **paranormal** if $\|Af\|^2 \leqslant \|f\| \|A^2 f\|$ for all $f \in \mathcal{D}(A^2)$.
- The following holds:

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- The theory of unbounded subnormal operators subsumes the theories of bounded subnormal operators and unbounded symmetric operators.
- bounded operators: Halmos (1950), Bram (1955), ... (two monographs by J. Conway).
- unbounded operators: Bishop (1957), Foiaş (1962), McDonald & Sundberg (1986), JS & Szafraniec (1985-89), ...

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The creation operator of quantum mechanics

- The **creation operator** a_+ is defined in $L^2(\mathbb{R})$ by

$$a_+ = \frac{1}{\sqrt{2}}\Big(x - \frac{d}{dx}\Big).$$

- The creation operator a_+ is subnormal.
- The creation operator a_+ is unitarily equivalent to the operator of multiplication by the independent variable "z" in the Segal-Bargmann space of entire functions that are square integrable with respect to the Gaussian measure on the complex plane [Segal, Bargmann 1961].

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Symmetric operators

- An operator S in $\mathcal{H}$ is said to be **symmetric** (resp., **selfadjoint**) if S is densely defined and $S \subseteq S^*$ (respectively, $S = S^*$).
- Clearly, selfadjoint operators are normal.
- A symmetric operator S in $\mathcal{H}$ is subnormal because it has a selfadjoint extension possibly in a larger Hilbert space [Naimark].
- S has a selfadjoint extension **within** $\mathcal{H}$ if and only if the deficiency indices of S are equal [von Neumann].

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- formally normal $\rightsquigarrow$ normal ?
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Formal normality and subnormality

- Formally normal operators **may not be** subnormal [Coddington 1965].
- There exist a **nonsubnormal** formally normal operator A and a polynomial $p \in \mathbb{C}[Z, \bar{Z}]$ **of degree** 3 such that $\mathcal{D}(A)$ is invariant for A and A^*, and

$$p(A, A^*)f = 0 \text{ for every } f \in \mathcal{D}(A);$$

  3 is the smallest possible degree [JS 1991].
- $\mathbf{p} = \mathbf{Y}(\mathbf{Y} - \mathbf{X}^2)$ where $\mathbf{X} = \frac{1}{2}(\mathbf{Z} + \bar{\mathbf{Z}})$ and $\mathbf{Y} = \frac{1}{2\mathrm{i}}(\mathbf{Z} - \bar{\mathbf{Z}})$.
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Stieltjes moment sequences

- A sequence $\{\gamma_n\}_{n=0}^{\infty}$ of nonnegative real numbers is called a **Stieltjes moment sequence** if there exists a (positive) Borel measure μ on $[0, \infty)$ such that

$$\gamma_n = \int_0^{\infty} x^n \, \mathrm{d}\,\mu(x), \quad n \geqslant 0;$$

  such a μ is called a **representing** measure of the sequence $\{\gamma_n\}_{n=0}^{\infty}$.
- We say that a Stieltjes moment sequence $\{\gamma_n\}_{n=0}^{\infty}$ is **determinate** if it has a unique representing measure; otherwise, we say that $\{\gamma_n\}_{n=0}^{\infty}$ is **indeterminate**.

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Generating Stieltjes moment sequences

- We say that an operator S in $\mathcal{H}$ **generates** Stieltjes moment sequences if the set $\mathcal{D}^\infty(S) := \bigcap_{n=0}^\infty \mathcal{D}(S^n)$ of all C^∞-vectors of S is dense in $\mathcal{H}$ and $\{\|S^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $f \in \mathcal{D}^\infty(S)$.

Theorem (Lambert 1976)

A bounded operator on $\mathcal{H}$ is subnormal if and only if it generates Stieltjes moment sequences.

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A question

- Lambert's theorem is not true for unbounded operators.
- Recall that there are nonsubnormal formally normal operators (hence hyponormal) which generate Stieltjes moment sequences.
- The question is whether there are closed nonhyponormal operators that generate Stieltjes moment sequences?

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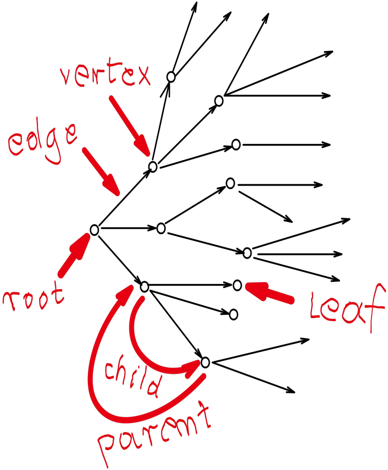
Directed trees

- $V^\circ = V \setminus \{\text{root}\}$ if $\mathscr{T}$ has a root and $V^\circ = V$ if $\mathscr{T}$ is rootless.
- $\mathscr{T} = (V, E)$ is a directed tree (V = vertices, E = edges).



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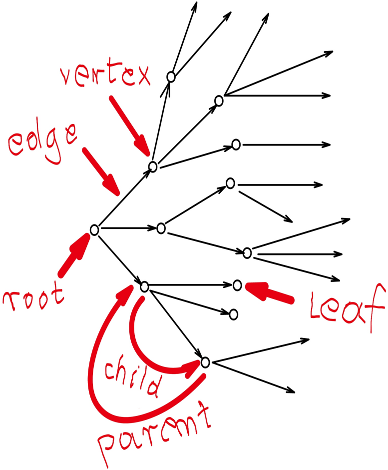

$\ell^2(V)$ over a directed tree

- $\ell^2(V)$ is the Hilbert space of square summable complex functions on V with the standard inner product

$$\langle f, g \rangle = \sum_{u \in V} f(u)\overline{g(u)}, \quad f, g \in \ell^2(V).$$

- For $u \in V$, we define $e_u \in \ell^2(V)$ by

$$e_u(v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

$\{e_u\}_{u \in V}$ is an orthonormal basis of $\ell^2(V)$.

- $\mathscr{E}_V =$ the linear span of the set $\{e_u \colon u \in V\}$.

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- Let $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$ be a system of complex numbers (called **weights**).
- Define the operator S_λ in $\ell^2(V)$ by

$$\mathcal{D}(S_\lambda) = \left\{ f \in \ell^2(V) \colon S_\lambda f \in \ell^2(V) \right\},$$

$$(S_\lambda f)(v) = \begin{cases} \lambda_v \cdot f\big(\mathrm{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \text{root}, \end{cases} \qquad f \in \mathcal{D}(S_\lambda).$$

- The operator S_λ is called a **weighted shift** on the directed tree $\mathscr{T}$ with weights λ [Jabłoński, Jung & JS 2012].
- Weighted adjacency operators [Fujii, Sasaoka & Watatani 1991]; weighted composition operators [Carlson 1990 - discrete and bounded], [Campbell & Hornor 1993 - bounded and partially unbounded].

Weighted shifts on directed trees

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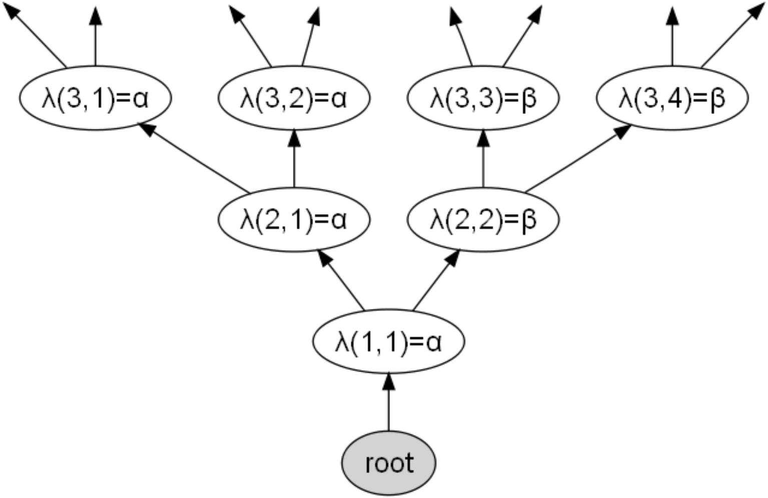
Figure



- Each weighted shift S_λ on a directed tree $\mathscr{T}$ is closed.
- A weighted shift S_λ is densely defined if and only if $\mathscr{E}_V \subseteq \mathcal{D}(S_\lambda)$.
- If S_λ is densely defined, then

$$S_\lambda e_u = \sum_{v \in \mathrm{Chi}(u)} \lambda_v e_v, \quad u \in V,$$

where $\mathrm{Chi}(u) = \{v \in V : (u, v) \in E\}$ is the set of all **children** (or **successors**) of the vertex u.

Elementary properties of w.s.'s on directed trees

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Hyponormality

Theorem (Budzyński, Jabłoński, Jung & JS 2012)

Let $S_{\boldsymbol\lambda}$ be a densely defined weighted shift on a leafless directed tree $\mathscr{T} = (V, E)$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then $S_{\boldsymbol\lambda}$ jest hyponormal if and only if

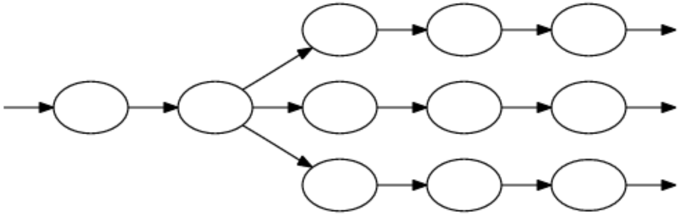
$$\sum_{v \in \mathrm{Chi}(u)} \frac{|\lambda_v|^2}{\|S_{\boldsymbol\lambda} e_v\|^2} \leqslant 1, \quad u \in V.$$

(Chi(u) *is the set of all children of* u).

The directed tree $\mathscr{T}_{\eta,\kappa}$.

$\eta \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ and $\kappa \in \{0, 1, 2, \ldots\} \cup \{\infty\}$.

$\mathscr{T}_{\eta,\kappa}$ is a directed tree with one branching vertex and η branches; its trunk consists of $\kappa + 1$ vertices (counting the branching vertex):



Towards a counterexample

- A nonhyponormal weighted shift S_λ that generates Stieltjes moment sequences will be constructed on the directed tree $\mathscr{T}_{\infty,\kappa}$.

General scheme

- Let $\{\gamma_n\}_{n=-\kappa}^{\infty} \subseteq (0,\infty)$ be a sequence such that

$$\gamma_0 = 1$$

$$\gamma_n = \int_0^\infty x^n \, d\,\nu(x), \quad n \in \mathbb{Z},\, n \geqslant -\kappa,$$

for some Borel measure ν on $\mathbb{R}_+$.

- Let ρ be a representing measure of the Stieltjes moment sequence $\{\gamma_{n+1}\}_{n=0}^{\infty}$ such that

$$0 < \int_0^\infty \frac{1}{x^n} \, d\,\rho(x) < \infty, \quad n \in J_{\kappa+1},$$

$$\mathrm{card}(\mathrm{supp}(\rho)) \geqslant \begin{cases} \eta & \text{if } \eta < \infty, \\ \aleph_0 & \text{if } \eta = \infty, \end{cases}$$

where $J_\iota = \{1,2,\ldots,\iota\} \setminus \{\infty\}$ for $\iota \in \{1,2,\ldots\} \cup \{\infty\}$.

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General scheme

- Let $\{\Omega_i\}_{i=1}^{\eta}$ be a sequence of pairwise disjoint Borel subsets of the interval $(0,\infty)$ such that

$$\rho(\Omega_i) > 0, \quad i \in J_\eta,$$
$$\bigsqcup_{i \in J_\eta} \Omega_i = (0,\infty).$$

  (Such a partition of the set $(0,\infty)$ always exists.)
- Define the sequence $\{\mu_{i,1}\}_{i\in J_\eta}$ of Borel probability measures on $\mathbb{R}_+$ by

$$\mu_{i,1}(\sigma) = \frac{1}{\rho(\Omega_i)}\rho(\Omega_i \cap \sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ i \in J_\eta.$$

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General scheme

- Next we define the family $\{\lambda_{i,j} \colon i \in J_\eta, j \in J_\infty\} \subseteq (0, \infty)$ by

$$\lambda_{i,j} = \begin{cases} \sqrt{\rho(\Omega_i)} & \text{for } j = 1, \\ \sqrt{\dfrac{\int_0^\infty x^{j-1} \, \mathrm{d}\, \mu_{i,1}(x)}{\int_0^\infty x^{j-2} \, \mathrm{d}\, \mu_{i,1}(x)}} & \text{for } j \geqslant 2, \end{cases} \quad i \in J_\eta.$$

- If $\kappa > 0$, then we define $\{\lambda_{-k}\}_{k=0}^{\kappa-1} \subseteq (0, \infty)$ by

$$\lambda_{-k} = \sqrt{\frac{\gamma_{-k}}{\gamma_{-(k+1)}}}, \quad k \in \mathbb{Z}_+, \, 0 \leqslant k < \kappa.$$

- Let S_λ be the weighted shift on $\mathscr{T}_{\eta,\kappa}$ with the above-defined weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$.
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Theorem (Jabłoński, Jung & JS 2012)

Under the above assumptions and definitions, we have

- $\mathcal{E}_{V_{\eta,\kappa}} \subseteq \mathcal{D}^\infty(S_\lambda)$,
- $\{\|S_\lambda^n f\|^2\}_{n=0}^\infty$ *is a Stieltjes moment sequence for every* $f \in \mathcal{D}^\infty(S_\lambda)$, *so* S_λ *generates Stieltjes moment sequences,*
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A counterexample - the directed tree $\mathscr{T}_{\infty,\kappa}$

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For every $\kappa \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ there exists an injective weighted shift S_λ on $\mathscr{T}_{\infty,\kappa}$ such that:

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♣ The proof of the above theorem depends heavily on some subtle properties of N-extremal measures. Recall that each indeterminate Stieltjes moment sequence possesses a continuum of N-extremal measures.

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Composition operators in L^2-spaces

- $(X, \mathcal{A}, \mu)$ is a σ-finite measure space.
- $\phi\colon X \to X$ is an $\mathcal{A}$**-measurable** transformation, i.e., $\phi^{-1}(\Delta) \in \mathcal{A}$ for every $\Delta \in \mathcal{A}$.
- If ϕ is **nonsingular**, i.e., the measure $\mu \circ \phi^{-1}$ given by $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$ for $\Delta \in \mathcal{A}$ is absolutely continuous with respect to μ, then the operator C_ϕ in $L^2(\mu)$ given by

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Subnormality of bounded composition operators

Theorem (Lambert 1988)

Let C_ϕ be a bounded composition operator on $L^2(\mu)$. Then the following two conditions are equivalent:

- *C_ϕ is subnormal,*
- *for μ-a.e. $x \in X$, $\{h_n(x)\}_{n=0}^{\infty}$ is a Stieltjes moment sequence,*

where $h_n \colon X \to [0, \infty]$ is the Radon-Nikodym derivative of the measure $\mu \circ (\phi^n)^{-1}$ with respect to μ.

(ϕ^0 is the identity mapping on X, $\phi^{n+1} = \phi \circ \phi^n$ for $n \geqslant 0$.)

- There are two more conditions characterizing the subnormality of bounded composition operators C_ϕ; however all of them are equivalent in the unbounded case.

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- *for μ-a.e. $x \in X$, $\{h_n(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence,*

  where $h_n\colon X \to [0,\infty]$ is the Radon-Nikodym derivative of the measure $\mu \circ (\phi^n)^{-1}$ with respect to μ.

  (ϕ^0 is the identity mapping on X, $\phi^{n+1} = \phi \circ \phi^n$ for $n \geqslant 0$.)

♣ There are two more conditions characterizing the subnormality of bounded composition operators C_ϕ; however all of them are equivalent in the unbounded case.

A problem

- Does Lambert's theorem remain true for unbounded composition operators in L^2-spaces?
- The answer is in the negative.
- Formally normal (in particular, symmetric) composition operators in L^2spaces are always normal.
- This means that there is no chance to adapt the methods used in general operator theory to the context of composition operators in L^2-spaces.

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Composition operators versus weighted shifts

Theorem (Jabłoński, Jung & JS 2012)

Let $S_{\boldsymbol\lambda}$ be a weighted shift on a rootless directed tree $\mathscr{T} = (V, E)$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Suppose V is countably infinite.

- *Then the operator $S_{\boldsymbol\lambda}$ is unitarily equivalent to a composition operator C in an L^2-space over a σ-finite measure space.*
- *Moreover, if the directed tree $\mathscr{T}$ is leafless, then C can be made injective.*

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A problem

- Find a criterion for subnormality of unbounded composition operators in L^2 spaces.
- It should cover the case of bounded composition operators.
- No restrictions on domains of powers of operators in question.
- The main difficulty: the known criteria for subnormality of general Hilbert space operators (Bishop, Foiaş, Szafraniec) do not help us to solve the problem.

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The conditional expectation

- We assume that the transformation ϕ is nonsingular and C_ϕ is densely defined.
- If $f\colon X \to \mathbb{R}_+$ is an $\mathcal{A}$-measure function, then there exists a unique (up to sets of μ-measure zero) $\phi^{-1}(\mathcal{A})$-measurable function $\mathsf{E}(f)\colon X \to \mathbb{R}_+$ such that

$$\int_{\phi^{-1}(\varDelta)} f \,\mathrm{d}\,\mu = \int_{\phi^{-1}(\varDelta)} \mathsf{E}(f) \,\mathrm{d}\,\mu, \quad \varDelta \in \mathcal{A}.$$

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The consistency condition

- $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ is said to be an $\mathcal{A}$**-measurable family of probability measures** if the set-function $P(x,\cdot)$ is a probability measure for every $x \in X$ and the function $P(\cdot,\sigma)$ is $\mathcal{A}$-measurable for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$.
- We say that an $\mathcal{A}$-measurable family of probability measures $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ satisfies the **consistency condition** if

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- If C_ϕ is subnormal, then C_ϕ is densely defined and injective.

Theorem (Budzyński, Jabłoński, Jung & JS 2013)

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and ϕ be a nonsingular transformation of X such that C_ϕ is densely defined and injective. Suppose there exists an $\mathcal{A}$-measurable family $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0, 1]$ of probability measures that satisfies the consistency condition. Then C_ϕ is subnormal.

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Generating moment sequences

- Find the relationship between the consistency condition and moments.

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A class of composition operators

- Let X be a nonempty set and $\phi\colon X \to X$ be a mapping. Set

$$E_\phi = \{(x, y) \in X \times X\colon x = \phi(y)\}.$$

  Then (X, E_ϕ) is a directed graph.
- Note that for every $y \in X$, $\phi(y)$ is the parent of y. Hence, $\phi^{-1}(\{x\})$ can be thought of as the set of all children of x.
- Connected directed graphs (X, E_ϕ) whose vertices, all but one, have valency one can be described explicitly.

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(X, E_ϕ) with one branching vertex

Theorem (Budzyński, Jabłoński, Jung & JS 2014)

Let (X, E_ϕ) be as above and let $\eta \in \{1, 2, 3, \dots\} \cup \{\infty\}$. Then the following two conditions are equivalent:

(i) *the directed graph (X, E_ϕ) is connected and there exists $\omega \in X$ such that $\operatorname{card}(\phi^{-1}(\{\omega\})) = \eta + 1$ and $\operatorname{card}(\phi^{-1}(\{x\})) = 1$ for every $x \in X \setminus \{\omega\}$,*

(ii) *(X, E_ϕ) takes one of the following two forms*:

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(ii-a) *there exist* $\kappa \in \{0, 1, 2, \ldots\}$ *and two disjoint systems* $\{x_i\}_{i=0}^{\kappa}$ *and* $\{x_{i,j}\}_{i=1}^{\eta}{}_{j=1}^{\infty}$ *of distinct points of* X *such that*

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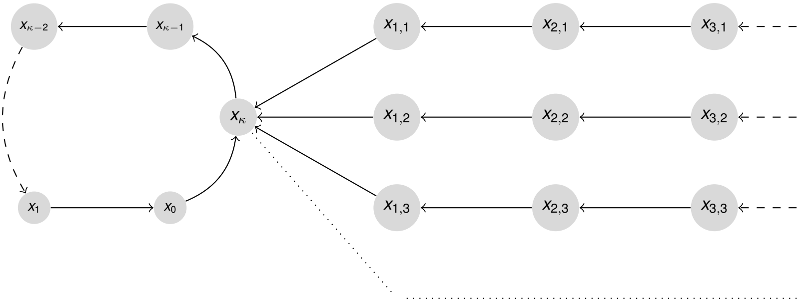

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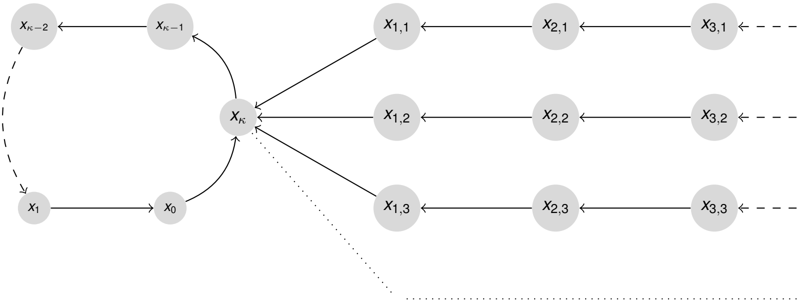

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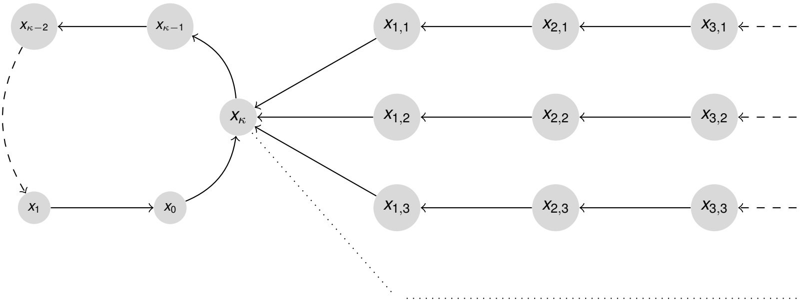

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- The class of composition operators C_ϕ in $L^2(X,\mu)$ with symbol ϕ as in (ii-b), where μ is a discrete measure[1] on X, coincides with the class of weighted shifts on the directed tree $\mathscr{T}_{\eta+1,\infty}$ with positive weights.
- Since the latter class was discussed earlier, we can concentrate on composition operators C_ϕ in $L^2(X,\mu)$ with symbol ϕ as in (ii-a), where μ is a discrete measures on X.

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Theorem (Budzyński, Jabłoński, Jung & JS 2014)

Let $X = X_{2,0}$ and $\phi = \phi_{2,0}$. Then there exists a (σ-finite) discrete measure μ on $(X, 2^X)$ such that

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$\mathcal{D}(S^2) = \{0\}$

- In 1940 Naimark gave a remarkable example of a closed symmetric operator S whose square has trivial domain, i.e.

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- As recently shown, symmetric weighted shifts on directed trees are automatically bounded. The same is true for composition operators in L^2-spaces.
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Necessity

- We say that a directed tree $\mathscr{T} = (V, E)$ is **extremal** if for every vertex $u \in V$, the set $\mathrm{Chi}(u)$ of all successors of u is countably infinite. Up to isomorphism, there are only two extremal directed trees, one with root, the other without.

Theorem (Jabłoński, Jung & JS 2014)

Let $\mathscr{T} = (V, E)$ be a directed tree such that $V^\circ \neq \varnothing$. Then the following conditions are equivalent:

(i) *there exists a family $\lambda = \{\lambda_v\}_{v \in V^\circ}$ of nonzero complex numbers such that $\overline{\mathcal{D}(S_\lambda)} = \ell^2(V)$ and $\mathcal{D}(S_\lambda^2) = \{0\}$,*

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- Using a criterion for subnormality of unbounded weighted shifts on directed trees due to BJJS, we can prove the following.

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Suppose $\mathscr{T} = (V, E)$ is an extremal directed tree and $n \in \{1, 2, 3, \ldots\}$. Then there exists a subnormal weighted shift S_λ on $\mathscr{T}$ such that S_λ^n is densely defined and $\mathcal{D}(S_\lambda^{n+1}) = \{0\}$.

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Sources

Talk is based on the following papers:

1. Z. J. Jabłoński, I. B. Jung, J. Stochel, Weighted shifts on directed trees, *Mem. Amer. Math. Soc.* **216** (2012), no. 1017, viii+107pp.

2. Z. J. Jabłoński, I. B. Jung, J. Stochel, A non-hyponormal operator generating Stieltjes moment sequences, *Journal of Functional Analysis* **262** (2012), 3946-3980.

3. P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Unbounded subnormal composition operators in L^2-spaces (arXiv:1303.6486), submitted, 44 pp.

4. P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Subnormality of composition operators in L^2 spaces over directed graphs with one circuit, work in progress, pp. $35+$.

5. P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Subnormal weighted shifts on directed trees whose nth powers have trivial domain.