# On the hyperinvariant subspace problem for asymptotically nonvanishing contractions

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Based on a joint work with L. Kérchy.

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$\mathcal{H}$ complex, separable Hilbert space.
$\mathcal{L}(\mathcal{H})$ bounded, linear operators on $\mathcal{H}$.
Non-trivial invariant subspace of $T \in \mathcal{L}(\mathcal{H})$: $\{0\} \neq \mathcal{M} \neq \mathcal{H}$, $T\mathcal{M} \subset \mathcal{M}$.
A non-trivial invariant subspace is hyperinvariant if $C\mathcal{M} \subset \mathcal{M}$ for all $C \in \{T\}'$.

ISP: Does every operator $T \in \mathcal{L}(\mathcal{H})$ have a non-trivial invariant subspace?
HSP: Does every non-scalar operator $T \in \mathcal{L}(\mathcal{H}) \setminus \mathbb{C}I$ have a non-trivial hyperinvariant subspace?

$||T|| \leq 1$

- $T = T_1 \oplus T_2$
  - $T_1$ is c.n.u.: $\nexists\, \mathcal{M} \in \operatorname{Lat} T_1 \colon T_1|\mathcal{M}$ is unitary
  - $T_2$ is unitary
  - $T$ is absolutely continuous if $T_2$ is a.c., i.e. if the spectral measure of $T_2$ is a.c. with respect to Lebesgue measure

Considering (ISP) and (HSP) we can suppose that $T$ is a.c.

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stable subspace:
$\mathcal{H}_0(T) = \{h \in \mathcal{H} : \lim_{n\to\infty} \|T^n h\| = 0\} \in \text{Hlat}\, T$

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## Definition

$(X, V)$ is a *unitary asymptote* of the contraction $T$, if

i) $V \in \mathcal{L}(\mathcal{K})$ unitary,

ii) $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ contractive, $XT = VX$,

iii) $\forall (X', V')$ satisfying (i) and (ii), $\exists! \, Y \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ contractive, $YV = V'Y$ and $X' = YX$.

Now such a pair exists, it is unique up to isomorphism, $\|Xh\| = \lim_{n\to\infty} \|T^n h\|$ for all $h \in \mathcal{H}$, and $V \in \mathcal{L}(\mathcal{K})$ is a.c.

## Definition

The measurable support of the spectral measure of $V$ is called the *residual set* $\omega(T)$ of $T$.

(unique up to sets of zero Lebesgue measure)

$\omega(T) \subset \sigma(V) \subset \sigma(T)$

$T \in \mathcal{L}(\mathcal{H})$ a.c. contraction

$\phi_T : H^\infty \to \mathcal{L}(\mathcal{H}), f \mapsto f(T)$

- weak-* continuous
- contractive: $||f(T)|| \leq ||f||_\infty$
- unital algebra-homomorphism
- $\chi(T) = T$

Uniquely determined:

Sz.-Nagy–Foias functional calculus for $T$.

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$\Phi_T$ is monotone:
$|f(z)| \leq |g(z)|$ for every $z \in \mathbb{D} \Rightarrow$
$\|f(T)x\| \leq \|g(T)x\|$ for every $x \in \mathcal{H}$.

In notation: $f \overset{a}{\prec} g \Rightarrow f(T) \overset{a}{\prec} g(T)$.

$F = \{f_n\}_{n=1}^{\infty} \subset H^\infty$ decreasing $\Rightarrow F(T) = \{f_n(T)\}_{n=1}^{\infty}$ is also decreasing, and the set

$$\mathcal{H}_0(T,F) = \left\{x \in \mathcal{H} : \lim_{n\to\infty} \|f_n(T)x\| = 0\right\} \in \mathrm{Hlat}\, T$$

$\varphi_F(\zeta) = \lim_{n\to\infty} |f_n(\zeta)|$ for a.e. $\zeta \in \mathbb{T}$

## Proposition

*If $\varphi_F(\zeta) > 0$ a.e. then $\mathcal{H}_0(T,F) = \{0\}$.*

$$N_F = \{\zeta \in \mathbb{T} : \varphi_F(\zeta) > 0\}$$

**Definition**

$T$ is *asymptotically non-vanishing on* $\alpha \in \mathcal{B}(\mathbb{T})$, if $\mathcal{H}_0(T, F) = \{0\}$ whenever $N_F \cap \alpha \neq \emptyset$.

**Definition**

The *quasianalytic spectral set* $\pi(T)$ of $T$ is the largest Borel set where $T$ is non-vanishing.

**Theorem (L. Kérchy (2001))**

*$\pi(T) \subset \omega(T)$, and if $\pi(T) \neq \omega(T)$ then* $\mathrm{Hlat}\, T$ *is non-trivial.*

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An a.c. contraction $T$ is called *quasianalytic* if $\pi(T) = \omega(T) \neq \emptyset$.

*Quasiaffine transform*: $A \prec B$, if there exists a quasiaffinity $Q$ (i.e. an injective transformation with dense range) such that $QA = BQ$.
*Quasisimilarity*: $A \sim B$, if $A \prec B$ and $B \prec A$.

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## Theorem (L. Kérchy)

*The operators $T$, $T_1$ and $T_2$ below are all a.c. contractions.*

1. *The unilateral shift $S \in \mathcal{L}(H^2)$ is quasianalytic with $\pi(S) = \mathbb{T}$.*
2. *If $T \prec S$, then $T$ is quasianalytic with $\pi(T) = \mathbb{T}$.*
3. *If $T$ is quasianalytic and $\mathcal{M}$ is a non-zero invariant subspace of $T$, then $T|\mathcal{M}$ is quasianalytic with $\pi(T|\mathcal{M}) = \pi(T)$.*
4. *If $T$ is quasianalytic and $f$ is a regular partially inner function satisfying the condition $\Omega(f) \cap \pi(T) \neq \emptyset$, then $f(T)$ is quasianalytic with $\pi(f(T)) = f(\Omega(f) \cap \pi(T))$.*
5. *If $T_1 \sim T_2$ and $T_1$ is quasianalytic, then $T_2$ is also quasianalytic with $\pi(T_2) = \pi(T_1)$.*

$T$ is a quasianalytic contraction $\Rightarrow T \in C_{10}$:
$\lim_{n\to\infty} \|T^{*n}h\| = 0 < \lim_{n\to\infty} \|T^n h\|$ for all $0 \neq h \in \mathcal{H}$.

$T \in \mathcal{L}(\mathcal{H})$ a.c. contraction, $(X, V)$ unitary asymptote.
$T$ is *asymptotically cyclic*, if $V \in \mathcal{L}(\mathcal{K})$ is cyclic, that is $\bigvee_{n=0}^{\infty} V^n y = \mathcal{K}$ holds for some $y \in \mathcal{K}$.

### Proposition

*If $T \in \mathcal{L}(\mathcal{H})$ is a contraction and $T \prec S$, then $T$ is asymptotically cyclic and $V|(X\mathcal{H})^- \cong S$.*

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- $\mathcal{L}_0(\mathcal{H})$: the set of asymptotically cyclic, quasianalytic contractions.
- $\mathcal{L}_1(\mathcal{H}) = \{T \in \mathcal{L}_0(\mathcal{H}) : \pi(T) = \mathbb{T}\}$

**Theorem**

*If $T \in \mathcal{L}_1(\mathcal{H})$ then*

1. $\vee \operatorname{Lat}_s T = \mathcal{H}$,
2. $\Phi_T$ *is an isometry,*
3. $H^\infty(T) = \mathcal{W}(T)$, *and*
4. $T$ *is reflexive.*

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*If $T \in \mathcal{L}(\mathcal{H})$ is a contraction such that $T \prec S$, then $T \in \mathcal{L}_1(\mathcal{H})$ and $H^\infty(T) = \{T\}'$.*

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*If $T \in \mathcal{L}_1(\mathcal{H})$, then $T|\mathcal{M} \in \mathcal{L}_1(\mathcal{M})$ holds for every non-zero invariant subspace $\mathcal{M}$ of $T$.*

## Theorem (L. Kérchy, V. Totik)

*For every $T_0 \in \mathcal{L}_0(\mathcal{H})$ we can find $T_1 \in \mathcal{L}_1(\mathcal{H})$ so that $T_0 T_1 = T_1 T_0$; hence $\{T_0\}' = \{T_1\}'$ and so $\mathrm{Hlat}\, T_0 = \mathrm{Hlat}\, T_1$.*

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*Let $T \in \mathcal{L}_1(\mathcal{H})$ be such that $\{T\}' \neq H^\infty(T)$. Then, for every $C \in \{T\}' \setminus H^\infty(T)$, we have $\mathrm{Lat}\, C \cap \mathrm{Lat_s}\, T = \emptyset$.*

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*For any $T \in \mathcal{L}_1(\mathcal{H})$, $\{T\}' = H^\infty(T)$ holds if and only if $\mathrm{Hlat}\, T = \mathrm{Lat}\, T$.*

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### Theorem

*Let $T \in \mathcal{L}_1(\mathcal{H})$ be such that $\{T\}' \neq H^\infty(T)$. Then the following statements are equivalent:*

1. $\mathrm{Hlat}\, T$ *is non-trivial;*
2. *there exists* $\mathcal{M} \in \mathrm{Lat_s}\, T$ *such that* $\vee\{C\mathcal{M} : C \in \{T\}'\} \neq \mathcal{H}$*;*
3. *there exists* $\mathcal{S} \subset \mathrm{Lat_s}\, T$ *such that* $\mathcal{H} \neq \vee \mathcal{S} \in \mathrm{Hlat}\, T$*.*

It is known that the unilateral shift $S$ is *cellular-indecomposable*, that is the intersection of any two non-zero invariant subspaces of $S$ is non-zero. The contraction $T \in \mathcal{L}_1(\mathcal{H})$ is called *quasiunitary*, if $X$ has dense range and so it is a quasiaffinity, where $(X, V)$ is a unitary asymptote of $T$.

## Proposition

*If $T \in \mathcal{L}_1(\mathcal{H})$, then the following conditions are equivalent:*

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If $T$ is not quasiunitary, then $\mathrm{Hlat}\, T = \mathrm{Lat}\, T$ is a rich lattice containing $\mathrm{Lat_s}\, T$ and $\left(\ker\left(T^* - \overline{\lambda} I\right)\right)^{\perp}$ $(\lambda \in \mathbb{D})$ because of $S^* \prec T^*$.

(HSP) in $\mathcal{L}_1(\mathcal{H})$ can be reduced to the quasiunitary case.

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