Lipschitz tensor product

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Abstract

We introduce the notion of a Lipschitz tensor product $X \boxtimes E$ of a pointed metric space X and a Banach space E. The concept of Lipschitz tensor product of elements of $X^\#$ and E^* yields the space $X^\# \boxtimes E^*$. To ensure the good behavior of a norm on $X \boxtimes E$ the concept of dualizable (respectively, uniform) Lipschitz cross-norm on $X \boxtimes E$ is defined. We study certain uniform dualizable Lipschitz cross-norms with good properties. In addition, we analyze the relationship between the Lipschitz tensor product and the injective and projective Banach-space tensor product between the Lipschitz-free space over X and E. In terms of the space $X^\# \boxtimes E^*$, we describe the spaces of Lipschitz compact (finite-rank, approximable) operators from X to E^*.

Definition and algebraic properties

Notation

- A map $f : X \to Y$ between metric spaces X and Y is said to be Lipschitz if
$$\mathrm{Lip}(f) = \sup\left\{\frac{d(f(x), f(y))}{d(x,y)} : x, y \in X,\ x \neq y\right\} < \infty.$$
- A pointed metric space X is a metric space with a base point in X denoted by 0.
- $\mathrm{Lip}_0(X, Y)$ denotes the set of all base-point preserving Lipschitz maps between two pointed metric spaces X and Y.
- We will consider a Banach space E over $\mathbb{K}$ as a pointed metric space with the zero vector as the base point, and denote its closed unit ball by B_E. E^* stands for the topological dual of E.
- $\mathrm{Lip}_0(X, E)$ is a Banach space with the norm $\mathrm{Lip}(\cdot)$. We designate $X^\# = \mathrm{Lip}_0(X, \mathbb{K})$.

Definition 1 (elementary Lipschitz tensors and Lipschitz tensor product)

For each $(x, y) \in X^2$, let $\delta_{(x,y)} : \mathrm{Lip}_0(X, E^*) \to E^*$ be the linear map given by
$$\delta_{(x,y)}(f) = f(x) - f(y) \qquad (f \in \mathrm{Lip}_0(X, E^*)).$$
For $(x, y) \in X^2$ and $e \in E$, the elementary Lipschitz tensor $\delta_{(x,y)} \boxtimes e : \mathrm{Lip}_0(X, E^*) \to \mathbb{K}$ is the bounded linear functional defined by
$$(\delta_{(x,y)} \boxtimes e)(f) = \delta_{(x,y)}(f)(e) = f(x)(e) - f(y)(e) \qquad (f \in \mathrm{Lip}_0(X, E^*)).$$
The Lipschitz tensor product $X \boxtimes E$ is defined as the vector subspace of $\mathrm{Lip}_0(X, E^*)^*$ spanned by the set $\left\{\delta_{(x,y)} \boxtimes e : (x, y) \in X^2,\ e \in E\right\}$.

Each element $u \in X \boxtimes E$ can be represented as $u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i$.

Theorem 2

i) $\langle X \boxtimes E, \mathrm{Lip}_0(X, E^*)\rangle$ forms a dual pair with bilinear form $\langle \cdot, \cdot\rangle$ associated given by
$$\langle u, f\rangle = u(f) = \sum_{i=1}^n (f(x_i) - f(y_i))(e_i), \quad \forall u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E,\ f \in \mathrm{Lip}_0(X, E^*).$$
ii) $X \boxtimes E$ is linearly isomorphic to the linear space $\mathcal{L}_F((X^\#, \tau_p); E)$ of all finite-rank linear operators from $X^\#$ into E which are continuous from the topology of pointwise convergence τ_p of $X^\#$ to the norm topology of E.

Definition 3 (Lipschitz tensor product functionals)

Let $g \in X^\#$ and $\phi \in E^*$. The Lipschitz tensor product functional of g and ϕ is the linear map $g \boxtimes \phi : X \boxtimes E \to \mathbb{K}$ given by
$$(g \boxtimes \phi)(u) = \sum_{i=1}^n (g(x_i) - g(y_i)) \cdot \phi(e_i), \quad \forall u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E.$$
The associated Lipschitz tensor product of $X \boxtimes E$ is the space $X^\# \boxtimes E^*$ defined as the linear subspace of the algebraic dual of $X \boxtimes E$ spanned by the set $\left\{g \boxtimes \phi : g \in X^\#,\ \phi \in E^*\right\}$.

A function $f \in \mathrm{Lip}_0(X, E^*)$ is said to have finite-rank if the subspace of E^* generated by $f(X)$, $\mathrm{lin}(f(X))$, is finite-dimensional.

Theorem 4

The associated Lipschitz tensor product $X^\# \boxtimes E^*$ is linearly isomorphic to the space of Lipschitz finite-rank functions from X to E^*.

Lipschitz cross-norms

Given $h \in \mathrm{Lip}_0(X, Y)$ and $T \in \mathcal{L}(E, F)$, where $\mathcal{L}(E, F)$ denotes the Banach space of all bounded linear operators between the Banach spaces E and F, we define the map $h \boxtimes T : X \boxtimes E \to Y \boxtimes F$ by
$$(h \boxtimes T)(u) = \sum_{i=1}^n \delta_{(h(x_i),h(y_i))} \boxtimes T(e_i), \quad \forall u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i \in X \boxtimes E.$$
The linear operator $h \boxtimes T$ is called the Lipschitz tensor product operator of h and T.

Definition 5

i) We say that a norm α on $X \boxtimes E$ is a Lipschitz cross-norm if
$$\alpha\left(\delta_{(x,y)} \boxtimes e\right) = d(x, y)\|e\|, \quad \forall (x, y) \in X^2,\ e \in E.$$
ii) A Lipschitz cross-norm α on $X \boxtimes E$ is said to be dualizable if for each $g \in X^\#$ and $\phi \in E^*$, it holds that $g \boxtimes \phi \in (X \boxtimes_\alpha E)^*$ and $\|g \boxtimes \phi\| \le \mathrm{Lip}(g)\|\phi\|$.

iii) A Lipschitz cross-norm α on $X \boxtimes E$ is called uniform if for each $h \in \mathrm{Lip}_0(X, X)$ and $T \in \mathcal{L}(E; E)$, it holds that $h \boxtimes T \in \mathcal{L}(X \boxtimes_\alpha E; X \boxtimes_\alpha E)$ and $\|h \boxtimes T\| \le \mathrm{Lip}(h)\|T\|$.

Proposition 6

If α is a dualizable Lipschitz cross-norm on $X \boxtimes E$, then $X^\# \boxtimes E^*$ is a linear subspace of $(X \boxtimes_\alpha E)^*$. Moreover, the canonical norm of operators on $X^\# \boxtimes E^*$, denoted by α', verifies the equality $\alpha'(g \boxtimes \phi) = \mathrm{Lip}(g)\|\phi\|$ for all $g \in X^\#$ and $\phi \in E^*$.

Definition 7

i) The Lipschitz injective norm on $X \boxtimes E$ is established, for each $u \in X \boxtimes E$, through
$$\varepsilon(u) = \sup\left\{|(g \boxtimes \phi)(u)| : g \in B_{X^\#},\ \phi \in B_{E^*}\right\}.$$
ii) The Lipschitz projective norm on $X \boxtimes E$ is set out, for each $u \in X \boxtimes E$, by
$$\pi(u) = \inf\left\{\sum_{i=1}^n d(x_i, y_i)\|e_i\| : u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes e_i\right\},$$
iii) Given $1 \le p \le \infty$, the Lipschitz p-nuclear norm on $X \boxtimes E$ is defined, for each $u \in X \boxtimes E$, by
$$d_p(u) = \inf\left\{\left\|(\lambda_1\delta_{(x_1,y_1)}, \ldots, \lambda_n\delta_{(x_n,y_n)})\right\|^{Lw}_{p'} \|(e_1, \ldots, e_n)\|_p : u = \sum_{i=1}^n \delta_{(x_i,y_i)} \boxtimes \lambda_i e_i\right\},$$
where p' is the conjugate index of p and
$$\left\|(\lambda_1\delta_{(x_1,y_1)}, \ldots, \lambda_n\delta_{(x_n,y_n)})\right\|^{Lw}_{p'} = \sup_{g \in B_{X^\#}}\left(\sum_{i=1}^n (|\lambda_i|\,|g(x_i) - g(y_i)|)^{p'}\right)^{1/p'} \quad \text{if } 1 \le p' < \infty,$$
$$\left\|(\lambda_1\delta_{(x_1,y_1)}, \ldots, \lambda_n\delta_{(x_n,y_n)})\right\|^{Lw}_{\infty} = \sup_{g \in B_{X^\#}}\left(\max\{|\lambda_i|\,|g(x_i) - g(y_i)| : 1 \le i \le n\}\right)$$

Theorem 8

i) The Lipschitz injective, projective and p-nuclear norms are uniform and dualizable Lipschitz cross-norms on $X \boxtimes E$.

ii) The Lipschitz projective norm π coincides with the canonical norm of operators induced in $X \boxtimes E$ by $\mathrm{Lip}_0(X, E^*)^*$, and it is the greatest Lipschitz cross-norm on $X \boxtimes E$.

iii) The Lipschitz injective norm ε is the least dualizable Lipschitz cross-norm on $X \boxtimes E$.

iv) A norm α on $X \boxtimes E$ is a dualizable Lipschitz cross-norm if and only if $\varepsilon \le \alpha \le \pi$.

Johnson's theorem rediscovered [5, Theorem 4.1]

$\mathrm{Lip}_0(X, E^*)$ is isometrically isomorphic to $(X \boxtimes_\pi E)^*$, via the isometric isomorphism $\Lambda : \mathrm{Lip}_0(X, E^*) \to (X \boxtimes_\pi E)^*$ given by $\Lambda(f)(u) = u(f)$ for all $f \in \mathrm{Lip}_0(X, E^*)$ and $u \in X \boxtimes E$.

Lipschitz approximable operators

Let us recall that a function $f \in \mathrm{Lip}_0(X, E)$ is said to be Lipschitz approximable if it is the limit in the Lipschitz norm Lip of a sequence of Lipschitz finite-rank functions from X to E.

Theorem 9

$X^\# \boxtimes_{\pi'} E^*$ is isometrically isomorphic to the space of Lipschitz finite-rank functions from X to E^*. As a consequence, the space of all Lipschitz approximable functions from X to E^* is isometrically isomorphic to the completion of $X^\# \boxtimes_{\pi'} E^*$.

We recall that a Banach space E is said to have the approximation property if given a compact set $K \subset E$ and $\varepsilon > 0$, there is a finite-rank bounded linear operator $T : E \to E$ such that $\|Te - e\| < \varepsilon$ for every $e \in K$. In [4, Corollary 2.5], it was shown that $X^\#$ has the approximation property if and only if the space of all Lipschitz approximable functions from X to E is the space of all Lipschitz compact functions from X to E. Using this fact, we derive the following result.

Corollary 10

Let X be a pointed metric space such that $X^\#$ has the approximation property. Then, for any Banach space E, the space of all Lipschitz compact functions from X to E^* is isometrically isomorphic to the completion of $X^\# \boxtimes_{\pi'} E^*$.

Lipschitz free space and Lipschitz tensor product

Let us recall that the Lipschitz-free Banach space over a pointed metric space X, $\mathcal{F}(X)$, is the closed linear subspace of $(X^\#)^*$ spanned by the set $\{\delta_x : x \in X\}$, where for each $x \in X$, δ_x is the evaluation functional at the point x defined on $X^\#$.

Theorem 11

i) The completion of $X \boxtimes_\varepsilon E$, $X \widehat{\boxtimes}_\varepsilon E$, is isometrically isomorphic to $\mathcal{F}(X) \widehat{\otimes}_\varepsilon E$.

ii) The completion of $X \boxtimes_\pi E$, $X \widehat{\boxtimes}_\pi E$, is isometrically isomorphic to $\mathcal{F}(X) \widehat{\otimes}_\pi E$.

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